K-Theory for Exceptional Extended Affine Weyl Groups

Bath – LMS Symposium on K-theory & Representation Theory – July 20, 2022

Nick Wright (joint work with G.A. Niblo & R. Plymen)
The Baum-Connes assembly map

Question: What is $K_\ast(C_r^\ast G)$?

$C_r^\ast G = \text{reduced group C}^\ast\text{-algebra}$
The Baum-Connes assembly map

Question: What is $K_\ast(C_r^*G)$?

$C_r^*G = \text{reduced group } C^\ast\text{-algebra} – \text{completion of } L^1(G) \text{ represented on } L^2(G)$
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Question: What is $K_* (C^*_r G)$?

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Conjecture (Baum-Connes)

The assembly map $K_*^G (EG) \to K_* (C^*_r G)$ is an isomorphism.

$EG$ is universal example of proper actions
Question: What is $K_*(C^*_rG)$?

$C^*_rG$ = reduced group $C^*$-algebra – completion of $L^1(G)$ represented on $L^2(G)$

**Conjecture (Baum-Connes)**

The assembly map $K^G_*(EG) \rightarrow K_*(C^*_rG)$ is an isomorphism.

$EG$ is universal example of proper actions

Geometry $\rightarrow$ Analysis
Example

(3,3,3) triangle group $G = \langle s_1, s_2, s_3 | s_i^2 = 1, (s_i s_j)^3 = 1 \rangle \cong \mathbb{Z}^2 \rtimes S_3$
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LHS = \( K_*^{\mathbb{Z}^2 \rtimes S_3 \mathbb{R}^2} \)
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$LHS = K^\mathbb{Z}^2 \rtimes S_3 (\mathbb{R}^2) = K^S_3 (\mathbb{T}^2)$
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\[ \text{LHS} = K_{\mathbb{Z}^2 \rtimes S_3}^* (\mathbb{R}^2) = K_{S_3}^* (\mathbb{T}^2) \]

\[ C_r^* G = (C_{* \mathbb{Z}^2}^*) \rtimes S_3 \approx C(\mathbb{T}^2) \rtimes S_3 \]
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$C_{r}^{*}G = (C^{*}\mathbb{Z}^2) \rtimes S_3$

$\cong C(\mathbb{T}^2) \rtimes S_3$

$\sim C_0(\mathbb{R}^2) \rtimes (\mathbb{Z}^2 \rtimes S_3)$

$= C_0(\mathbb{T}^2) \rtimes (\mathbb{Z}^2 \rtimes S_3)$
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The Baum-Connes isomorphism is $K_{S_3}^{S_3} (T) \cong K_{S_3}^{S_3} (T^\vee)$ where $T, T^\vee$ are dual tori
Langlands duality

\( \mathcal{G} \) a compact connected semi-simple Lie group

\( T \) a maximal torus with Lie algebra \( t \)
Langlands duality

\[ G \] a compact connected semi-simple Lie group

\[ T \] a maximal torus with Lie algebra \( \mathfrak{t} \)

\((X^*, \Phi, X_*, \Phi^\vee)\) is the root datum of \( G \)

\[ T = \mathfrak{t}/X_*, \; X^* = \text{dual lattice} \]

\[ \Phi = \text{roots, } \Phi^\vee = \text{coroots} \]
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**Definition**

The (real) Langlands dual of $G$ is the Lie group with datum $(X_*, \Phi^\vee, X^*, \Phi)$. 
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In the \( A_n, D_n, E_n \) cases \( \Phi^\vee = \Phi \)

The key difference is the lattices.

\[ \begin{array}{c}
A_n & B_n & C_n & D_n & E_6 & E_7 & E_8 & F_4 & G_2 \\
\begin{tikzpicture}
  \begin{scope}
    \foreach \x in {0,1,2,3,4,5,6}
    {\node [circle, fill] (n\x) at (\x,0) {};
     \node [circle, fill] (n\x+7) at (\x,1) {};
    }
    \draw (n0) -- (n1) -- (n2) -- (n3) -- (n4) -- (n5) -- (n6);
    \end{scope}
\end{tikzpicture}
\end{array} \]
Affine Weyl groups

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The extended affine Weyl group is

\[ W'_a(G) = X_* \rtimes W \]
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\( G = SU_3 \)

\( T = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha \beta \gamma = 1 \right\} \)
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$$W = S_3$$ acting by permutations

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\[ T^\vee = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ \end{pmatrix} \right\} / \{\delta I\} \]

\[ X^* = \left\{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\} \]
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$W'_a(G^\vee)$ is ‘cone’ group
Theorem (Niblo-Plymen-W)

For $G$ a compact connected semi-simple Lie group there is a commutative diagram:

$$
\begin{array}{ccc}
K^*_{\alpha} (G) (t) & \xrightarrow{\text{Baum Connes}} & K^*_* (C^r_{\alpha} (G)) \\
\downarrow & & \downarrow \\
K^* (C_0 (t) \rtimes W_{\alpha} (G)) & \xrightarrow{\text{Poincaré duality in KK}} & K^*_* (C_0 (t) \rtimes W_{\alpha} (G^\vee))
\end{array}
$$
Extended quotients and $K$-theory

By the equivariant Chern character of Baum and Connes the $K$-theory $K_*(C_0(t) \rtimes W'_a)$ is given (up to torsion) by the cohomology of the extended quotient $t//W'_a$.

**Definition**

The inertia space for $G$ acting on $X$ is

$$I(G, X) = \{(g, x) : gx = x\}.$$ 

The extended quotient is

$$X//G = I(G, X)/G.$$
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The action on $I(G, X) \subseteq G \times X$ is $h \cdot (g, x) = (g^h, hx)$.

It suffices to consider $g$ ranging over conjugacy class representatives.

$$X//G = \bigsqcup_{g \text{ c.c. rep.}} X^g/Z(g).$$
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Note: $t//W'_a = T//W$.  

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The $A_{n-1}$ case

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$PSU_n$ is the adjoint type group of type $A_{n-1}$ (trivial centre)
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For any $k|n$ the cyclic group $C_k$ of $k$th roots of unity is in the center of $SU_n$

$SU_n/C_k$ is a group of type $A_n$ – its Langlands dual is $SU_n/C_{k’}$ where $k’ = k/n$. 
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Theorem (Niblo-Plymen-W)

Let $T_k$ be the maximal torus in $SU_n/C_k$ (where $k|n$). Let $W$ be the Weyl group.

\[ T_k/W \text{ is homotopy equivalent to } \prod_{\mu} \mathbb{T}^{b(\mu)-1} \times Y_{\mu} \]

where $\mu$ ranges over partitions of $n$

$b(\mu)$ is the number of distinct parts of $\mu$

$Y_{\mu}$ is a discrete set of cardinality $\frac{\gcd(\mu)}{a} \sum_{s=0}^{a-1} \gcd(a, s)$ with $a = \gcd \left( \gcd(\mu), \frac{n}{\gcd(\mu)}, k, n/k \right)$
Question

What is $T//W$ where $T$ is a maximal torus and $W$ is the Weyl group of type $E_6$?
Exceptional type $E_6$

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We need to determine:

- Conjugacy class representatives
- Centralisers of these
- Fixed sets
Exceptional type $E_6$

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**Theorem (Carter 1972)**

*Let $W$ be the Weyl group of a simple Lie algebra. Conjugacy classes in $W$ are determined by **admissible diagrams**.*

In particular Carter enumerates conjugacy classes and computes the number of elements.
Roots in $E_6$

$E_6$ has 72 roots.

$\langle r, r \rangle = 2$ for all $r \implies r^\vee = r$ and $\langle r, r' \rangle = \pm 1$ or 0 for $r \neq r'$.
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Each connected subdiagram of the Dynkin diagram gives a root e.g.

$$r_T = r_2 + r_3 + r_4 + r_6$$
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Define $r_0 = -\max \text{ root}$
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Conjugacy class representatives

Write $s_i$ for the reflection given by root $r_i, \ i = 0, \ldots, 6.$
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| Type | CC Rep | CC Size | $|Z_W(w)|$ | Elementary Part | Index in $|Z_W(w)|$ |
|------|--------|---------|-----------|-----------------|-------------------|
| $\emptyset$ | $e$ | 1 | 51840 | $W$ | 1 |
| $A_1$ | $s_0$ | 36 | 1440 | $\langle s_0 \rangle \times \langle s_1, \ldots, s_5 \rangle$ | 1 |
| $A_1^2$ | $s_0s_1$ | 270 | 192 | $\langle s_0, s_1 \rangle \times \langle s_3, \ldots, s_5 \rangle$ | 2 |
| $A_2$ | $s_0s_6$ | 240 | 216 | $\langle s_0s_6 \rangle \times \langle s_1, s_2, s_4, s_5 \rangle$ | 2 |
| $A_3$ | $s_0s_1s_5$ | 540 | 96 | $\langle s_0, s_1, s_5 \rangle \times \langle s_3 \rangle$ | 6 |
| $A_2 \times A_1$ | $s_0s_6s_1$ | 1440 | 36 | $\langle s_0s_6, s_1 \rangle \times \langle s_4, s_5 \rangle$ | 1 |
| $A_3$ | $s_0s_6s_3$ | 1620 | 32 | $\langle s_0s_6s_3 \rangle \times \langle s_1, s_5 \rangle$ | 2 |
| $A_4^2$ | $s_0s_1s_5s_3$ | 45 | 1152 | $\langle s_0, s_1, s_5, s_3 \rangle$ | 72 |
| $A_2 \times A_2^2$ | $s_0s_6s_1s_5$ | 2160 | 24 | $\langle s_0s_6, s_1, s_5 \rangle$ | 2 |
| $A_2^2$ | $s_0s_6s_1s_2$ | 480 | 108 | $\langle s_0s_6, s_1s_2 \rangle \times \langle s_4, s_5 \rangle$ | 2 |
| $A_3 \times A_1$ | $s_0s_6s_3s_1$ | 3240 | 16 | $\langle s_0s_6s_3, s_1 \rangle \times \langle s_5 \rangle$ | 1 |
| $A_4$ | $s_0s_6s_3s_4$ | 5184 | 10 | $\langle s_0s_6s_3s_4 \rangle \times \langle s_1 \rangle$ | 1 |
| $D_4$ | $s_1s_5s_0T$ | 1440 | 36 | $\langle s_1s_5s_0T \rangle$ | 6 |
| $D_4[a_1]$ | $s_1Ts_5s_0^T$ | 540 | 96 | $\langle s_1Ts_5s_0^T \rangle$ | 24 |
| Type       | CC Rep                        | CC Size | $|Z_W(w)|$ | Elementary Part                          | Index in $|Z_W(w)|$ |
|------------|-------------------------------|---------|---------|------------------------------------------|------------------|
| $A_2^2 \times A_1$ | $s_0 s_6 s_5 s_1 s_2$         | 1440    | 36      | $\langle s_0 s_6, s_5, s_1 s_2 \rangle$ | 2                 |
| $A_3 \times A_1^2$ | $s_0 s_6 s_3 s_1 s_5$         | 540     | 96      | $\langle s_0 s_6 s_3, s_1, s_5 \rangle$ | 6                 |
| $A_4 \times A_1$   | $s_0 s_6 s_3 s_4 s_1$         | 5184    | 10      | $\langle s_0 s_6 s_3 s_4, s_1 \rangle$ | 1                 |
| $A_5$              | $s_0 s_6 s_3 s_4 s_5$         | 4320    | 12      | $\langle s_0 s_6 s_3 s_4 s_5 \rangle \times \langle s_1 \rangle$ | 1                 |
| $D_5$              | $s_0 s_6 s_3 s_4 s_3^{s_2 s_4}$ | 6480    | 8       | $\langle s_0 s_6 s_3 s_4 s_3^{s_2 s_4} \rangle$ | 1                 |
| $D_5[a_1]$         | $s_0 s_6 s_3 s_4 s_3^T$       | 4320    | 12      | $\langle s_0 s_6 s_3 s_4 s_3^{s_2 s_4 s_6} \rangle$ | 1                 |
| $A_2^3$            | $s_0 s_6 s_1 s_2 s_5 s_4$     | 80      | 648     | $\langle s_0 s_6, s_1 s_2, s_5 s_4 \rangle$ | 24                |
| $A_5 \times A_1$   | $s_0 s_6 s_3 s_4 s_5 s_1$     | 1440    | 36      | $\langle s_0 s_6 s_3 s_4 s_5, s_1 \rangle$ | 3                 |
| $E_6$              | $s_1 s_2 s_3 s_4 s_5 s_6$     | 4320    | 12      | $\langle s_1 s_2 s_3 s_4 s_5 s_6 \rangle$ | 1                 |
| $E_6[a_1]$         | $s_1 s_2 s_3 s_4 s_5 s_3^{s_3}$ | 5760    | 9       | $\langle s_1 s_2 s_3 s_4 s_5 s_3^{s_3} \rangle$ | 1                 |
| $E_6[a_2]$         | $s_6 s_2 s_0^T s_1^T s_4 s_3$ | 720     | 72      | $\langle s_6 s_2 s_0^T s_1^T s_4 s_3 \rangle$ | 12                |
The roots $r_0, r_3, r_2 + r_3 + r_4, r_1 + r_2 + r_3 + r_4 + r_5$ are orthogonal so define commuting reflections.
PSymmetries of the roots

The roots $r_0, r_3, r_2 + r_3 + r_4, r_1 + r_2 + r_3 + r_4 + r_5$ are orthogonal so define commuting reflections.

The product $u_1 := s_0s_3s_{r_2+r_3+r_4}s_{r_1+r_2+r_3+r_4+r_5}$ is an involution s.t.

$$u_1r_i = \begin{cases} -r_{6-i} & i=1,2,\ldots,5 \\ -r_i & i = 0, 3, 6 \end{cases}$$

Hence $s^u_1 = \begin{cases} s_{6-i} & i=1,2,\ldots,5 \\ s_i & i = 0, 3, 6 \end{cases}$
The roots \( r_0, r_3, r_2 + r_3 + r_4, r_1 + r_2 + r_3 + r_4 + r_5 \) are orthogonal so define commuting reflections.

The product \( u_1 := s_0 s_3 s_{r_2+r_3+r_4}s_{r_1+r_2+r_3+r_4+r_5} \) is an involution s.t.

\[
u_1 r_i = \begin{cases} -r_{6-i} & i=1,2,\ldots,5 \\ -r_i & i = 0, 3, 6 \end{cases}
\]

Hence \( s_i^{u_1} = \begin{cases} s_{6-i} & i=1,2,\ldots,5 \\ s_i & i = 0, 3, 6 \end{cases} \)

Similarly we can define \( u_2 \) and \( u_3 = u_2^{u_1} \) giving the other two symmetries.

\( \langle u_1, u_2 \rangle \) is the dihedral group \( D_3 \).
Symmetries of cc representatives

Example

$A_1^3$ case: $w = s_0 s_1 s_5$.

The elementary centraliser $\langle s_0, s_1, s_5, s_3 \rangle$ has index 6 in $Z_W(w)$. $\langle u_1, u_2 \rangle$ permutes $s_0, s_1, s_5$ hence

$$\langle u_1, u_2 \rangle < Z_W(w)$$

$u_1, u_2$ fix $s_3$ so

$$Z_W(w) = \langle s_0, s_1, s_5 \rangle \rtimes \langle u_1, u_2 \rangle \times \langle s_3 \rangle.$$
Symmetries of cc representatives

Example

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The elements \(u_1, u_2, u_3\) account for the remainder of the centraliser in cases

\(A_1^2, A_2, A_3, A_2 \times A_1, A_3, \ldots, A_2 \times A_1, D_4, A_2^2 \times A_1\)
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The elementary centraliser

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The elements $u_1, u_2, u_3$ account for the remainder of the centraliser in cases

$A^2_1, A_2, A^3_1, A_3, A_2 \times A^2_1, A^2_2, D_4, A^2_2 \times A_1$

Example

$A^4_1$ case: $w = s_0 s_1 s_5 s_3$.

One notes that $u_1, u_2, T \in Z_W(w)$.

$\langle s_0, s_1, s_5, T \rangle$ is a Weyl group of type $D_4$ and

$Z_W(w) = W(D_4) \rtimes D_3$
Complex reflections

$A_2^3$ case: $w = s_0s_6s_1s_2s_5s_4$.

$w^3 = 1$ and $w$ has eigenvalues $e^{\pm \frac{2}{3} \pi i}$
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The $\zeta = e^{\frac{2}{3} \pi i}$ eigenspace is spanned by

$$\{r_1 - \zeta r_2, r_5 - \zeta r_4, r_0 - \zeta r_6\}$$

The sum of these is $(\zeta^2 - 1)(r_T - \zeta r_3)$
which is the $\zeta$ eigenvector of the rotation $Ts_3$. 
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Similarly a $\overline{\zeta}$ eigenvector of $T_s_3$ is in the $\overline{\zeta}$ eigenspace of $w$ so $T_s_3 \in Z_W(w)$. 
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Lemma

The group $\langle s_0 s_6, Ts_3, s_5 s_4 \rangle$ is the complex reflection group $G_{25} = 3 \overline{3} 3$.
Complex reflections

$A_2^3$ case: $w = s_0 s_6 s_1 s_2 s_5 s_4$.

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**Lemma**

The group $\langle s_0 s_6, T_{s_3}, s_5 s_4 \rangle$ is the complex reflection group $G_{25} = \bullet \bullet \bullet$.

Sketch: Let $J = \frac{1}{\sqrt{3}}(2w + I)$.

$w^2 + w + I = 0$ so $J^2 = -I$.

This makes $t$ a $\mathbb{C}$-vector space and $s_0 s_6, T_{s_3}, s_5 s_4$ are complex reflections.
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The group $\langle s_0s_6, Ts_3, s_5s_4 \rangle$ is the complex reflection group $G_{25}$.

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This makes $t$ a $\mathbb{C}$-vector space and $s_0s_6, Ts_3, s_5s_4$ are complex reflections.

An $\mathbb{R}$-linear map of $t$ commutes with $w$ iff it is $\mathbb{C}$-linear.

The generators satisfy the braid relations so give $G_{25}$. 

Centralisers as reflection groups

Theorem (Springer, 1974)

For $W$ a Weyl group, if $w \in W$ is regular then $Z_W(w)$ is a complex reflection group.

Regular: $\exists$ regular eigenvector of $w$ in $t \otimes \mathbb{C}$

$\langle s_1, s_2 \rangle \rtimes \langle u_1 \rangle$ is a reflection group over $\mathbb{F}_3$. 
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General context: $\mathcal{G}$ compact connected semisimple Lie group.

Fixed sets are subgroups of maximal torus $T = t/X_*$
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Let $\ell(w)$ denote the word length of $w$ with respect to all root reflections in $W$.

Lemma (Carter)

$\ell(w)$ is the number of eigenvalues of $w$ on $t$ which are $\neq 1$. ($= \text{rank}(I - w)$)

Hence identity component of $T^w$ is $T^{n-\ell(w)}$ where $n = \text{rank}$. 
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There is a faithful pairing $F_w \times F_w^\vee \to U(1)$. (i.e $F_w^\vee \cong \hat{F_w}$)
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There is a faithful pairing $F_w \times F_w^\vee \to U(1)$. (i.e $F_w^\vee \cong \hat{F}_w$)

Moreover the actions of $Z_W(w)$ on $F_w$ and $F_w^\vee$ are dual.
Comparing Fixed sets

Let $T, T^\vee$ be maximal tori in Langlands dual groups.

The identity component in each case is $\pi_0(T^w)$ is dual to $\pi_0((T^\vee)^w)$ so they are isomorphic.

Theorem (NPW)

For any $G$ and $w \in W(G)$ there is a (noncanonical) isomorphism $T^w \cong (T^\vee)^w$.

Theorem (PNW)

Let $g$ be the g.c.d. of determinants of $\ell^w$-minors of $I^w$ using coordinates with respect to a basis of $X^\ast$. Then $|F^w| = |\hat{F}^w| = g$.
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Theorem (NPW)
## Sectors of Extended Quotient for $E_6$ (simply connected)

<table>
<thead>
<tr>
<th>Type</th>
<th>Fixed set</th>
<th>Quotient</th>
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<tr>
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<td>$\Delta^6$</td>
</tr>
<tr>
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<td>$\Delta^3 \times S^1$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$T^4$</td>
<td>$\text{SP}^2(\Delta^2)$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$T^3$</td>
<td>$\Delta^2 \times \Delta^1$</td>
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<td>$T^3 \times T^1$</td>
<td>$\Delta^2 \cup \Delta^2$</td>
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<tr>
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<td>$T^3$</td>
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<td>$T^2 \times V_4$</td>
<td>$\text{SP}^2(T^1)$</td>
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<td>$T^2$</td>
<td>$\Delta^2$</td>
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<td>Four points</td>
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The Hessian group is the group of determinant 1 affine transformations of $\mathbb{F}_3^2$.

This is $G_{25}/C_3$.
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Of course $w$ acts trivially on the fixed set so we have action of Hessian group $G_{25}/\langle w \rangle$. 
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The action on $\mathbb{F}_3^3$ consists of linear maps of $\mathbb{F}_3^3$ fixing one direction.
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The action of $Z_W(w)/\langle w \rangle$ on $\mathbb{F}_3^3$ is the dual of the standard Hessian representation – it acts transitively on the non-central points.
Sectors of Extended Quotient for $E_6$ (adjoint type)

For each $w \in W$ there are 2 possibilities

- $Z$ is in the identity component

Theorem

For all $w \in W$ the sectors $T_w/Z/_{W(w)}$ and $(T \lor w)/Z/_{W(w)}$ are homotopy equivalent.

Conjecture

For any compact connected semisimple Lie group the sectors $T_w/Z/_{W(w)}$ and $(T \lor w)/Z/_{W(w)}$ are homotopy equivalent $\forall w$. 

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  $$ (T^\vee)^w/Z_W(w) = (T^w/Z_W(w))/Z $$

- $Z \hookrightarrow \pi_0(T^w)$
  
  $$ \Rightarrow \text{identity components of } T^w, (T^\vee)^w \text{ are equal} $$

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For all $w \in W$ the sectors $T^w/Z_W(w)$ and $(T^\vee)^w/Z_W(w)$ are homotopy equivalent.

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  $(T^\vee)^w / Z_W(w) = (T^w / Z_W(w)) / Z$

- $Z \hookrightarrow \pi_0(T^w)$
  $\implies$ identity components of $T^w$, $(T^\vee)^w$ are equal
  $T^w / Z$ has index 3 in $(T^\vee)^w$
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  \[ (T^\vee)^w = T^w/Z \text{ and } \pi_0(T^w) \xrightarrow{\cong} \pi_0((T^\vee)^w) \]
  \[ (T^\vee)^w/Z_W(w) = (T^w/Z_W(w))/Z \]

- $Z \hookrightarrow \pi_0(T^w)$
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  \implies \text{identity components of } T^w, (T^\vee)^w \text{ are equal}
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  \[ T^w/Z \text{ has index 3 in } (T^\vee)^w \]

**Theorem**

*For all $w \in W$ the sectors $T^w/Z_W(w)$ and $(T^\vee)^w/Z_W(w)$ are homotopy equivalent.*
Sectors of Extended Quotient for $E_6$ (adjoint type)

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**Theorem**

For all $w \in W$ the sectors $T^w/Z_W(w)$ and $(T^\vee)^w/Z_W(w)$ are homotopy equivalent.

**Conjecture**

For any compact connected semisimple Lie group the sectors $T^w/Z_W(w)$ and $(T^\vee)^w/Z_W(w)$ are homotopy equivalent $\forall w$. 
Applications: Iwahori-spherical block

\[ G_p = \text{split adjoint group over } F = \mathbb{Q}_p \text{ with } E_6 \text{ root system.} \]

\[ T_F = \text{maximal torus of } G_p, \quad ^0 T_F \subset T_F = \text{maximal compact.} \]
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The *Iwahori-spherical block* $= \text{irreducible smooth } G \text{-representations given by parabolic induction from unramified characters.}$
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**Theorem (Aubert-Baum-Plymen-Solleveld)**

\textit{For } \( p \neq 2, 3, 5 \text{ there is cts bijection } T//W \rightarrow \text{tempered representations in Iwahori-spherical block} \)
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**Theorem (Aubert-Baum-Plymen-Solleveld)**

*For \( p \neq 2, 3, 5 \) there is cts bijection \( T//W \to \text{tempered representations in Iwahori-spherical block} \)*

Our calculation gives geometric structure of tempered reps in Iwahori-spherical block.
Applications: K-theory

**Theorem**

Let $W'_a$ be the extended affine Weyl group of a Lie group of type $E_6$. Up to torsion $K_*(C^*_rW'_a)$ is

\[
\begin{cases}
\mathbb{Z}^{47} & \text{in dimension 0} \\
\mathbb{Z}^{11} & \text{in dimension 1}
\end{cases}
\]

This holds for both simply connected and adjoint type cases.
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Questions?