# Complex semisimple quantum groups, deformations, and the Baum-Connes assembly map

Christian Voigt

University of Glasgow
christian.voigt@glasgow.ac.uk
http://www.maths.gla.ac.uk/~cvoigt/index.xhtml

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# Baum-Connes

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More generally, the *Baum-Connes conjecture with coefficients* states that

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What happens if $G$ is a locally compact *quantum* group?

# Quantum groups

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The theory of locally compact quantum groups started from attempts to generalise Pontrjagin duality from the case of locally compact abelian groups to arbitrary locally compact groups.

If $G$ is a locally compact abelian group then the Pontrjagin dual $\hat{G}$ of all unitary characters $\chi$ is a locally compact abelian group in a natural way, and there is a canonical isomorphism

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If $G$ is not abelian this does not work any longer, and a good replacement for $\hat{G}$ is the full/reduced group $C^*$-algebra of $G$.

# Locally compact quantum groups

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## Definition

A *Hopf-$C^*$-algebra* is a $C^*$-algebra $H$ together with a nondegenerate injective $*$-homomorphism $\Delta : H \to M(H \otimes H)$ such that

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & M(H \otimes H) \\ \downarrow & & \downarrow \scriptstyle{\mathrm{id} \otimes \Delta} \\ M(H \otimes H) & \xrightarrow{\Delta \otimes \mathrm{id}} & M(H \otimes H \otimes H) \end{array}$$

is commutative and $\Delta(H)(1 \otimes H)$ and $(H \otimes 1)\Delta(H)$ are dense subspaces of $H \otimes H$.

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A *locally compact quantum group* is given by a Hopf-$C^*$-algebra $H$ together with *left and right Haar weights*.

# Locally compact quantum groups

### Example

- If $G$ is a locally compact group then $H = C_0(G)$ defines a locally compact quantum group.
  The comultiplication $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G)) = C_b(G \times G)$ is given by
  $$\Delta(f)(s,t) = f(st),$$
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  and the integrals are given by left/right Haar measure.
- The group $C^*$-algebra $\hat{H} = C^*_r(G)$ defines a locally compact quantum group via $\hat{\Delta} : C^*_r(G) \to M(C^*_r(G) \otimes C^*_r(G))$ given by
  $$\hat{\Delta}(u_t) = u_t \otimes u_t$$
  for all $t \in G$.

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  and view these algebras as different realizations of the same "quantum group $G$".
- In this notation, $C_0^{\mathrm{r}}(\hat{G}) = C_{\mathrm{r}}^*(G)$ where $\hat{G}$ is the *dual quantum group*.
- Pontrjagin duality holds: for any locally compact quantum group $G$, the dual $\hat{G}$ is again a locally compact quantum group, and we have a canonical isomorphism $\hat{\hat{G}} \cong G$.

# Complex quantum groups

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Complex quantum groups are locally compact quantum groups obtained as deformations of complex semisimple Lie groups.

Very roughly, given a complex group like $G = SL(n, \mathbb{C})$, one can deform its defining relations depending on a positive real parameter $q$, to obtain a quantum group $G_q$.

In the limit $q \to 1$, this construction converges to the classical group $G$ in a certain sense.

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Complex quantum groups have links with

- Operator $K$-theory and the Baum-Connes conjecture
- Property (T) for $C^*$-tensor categories and subfactors
- Chern-Simons theory with complex gauge groups

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A little bit of history:

- Podleś-Woronowicz (1990) construct complex semisimple quantum groups on the $C^*$-algebra level.
- Pusz (1993), Pusz-Woronowicz (1994, 2000) completely classify the irreducible unitary representations of $SL_q(2, \mathbb{C})$.
- Buffenoir-Roche (1999) determine the Plancherel formula for $SL_q(2, \mathbb{C})$.
- Arano (2014, 2016) completely classifies the irreducible unitary representations of $SL_q(n, \mathbb{C})$, and most of the full unitary dual in general.

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- Start from the *Iwasawa decomposition* $G = KAN$.
- For the compact part $K$ there exists a deformation $K_q$ obtained using *quantized enveloping algebras*.
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- Start from the *Iwasawa decomposition* $G = KAN$.
- For the compact part $K$ there exists a deformation $K_q$ obtained using *quantized enveloping algebras*.
- According to the *quantum duality principle*, a quantization of $AN$ is given by the *Pontrjagin dual* $\hat{K}_q$ of $K_q$.
- The complex quantum group $G_q$ is the *quantum double*

$$G_q = K_q \bowtie \hat{K}_q.$$

We shall now explain these ingredients in more detail.

# Some notation

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Let $\mathfrak{g}$ be the Lie algebra of the complex semisimple group $G$.

- Let $q = e^h$ for some real nonzero number $h$.
- Let $N$ be the rank of $\mathfrak{g}$, and denote by $(a_{ij}) \in M_N(\mathbb{Z})$ the Cartan matrix.
- Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\mathfrak{t} \subset \mathfrak{h}$ correspond to the maximal torus $T \subset K$
- Let $\Delta = \Delta^+ \cup \Delta^-$ be the root system with simple roots $\alpha_1, \dots, \alpha_N \subset \mathfrak{h}^*$.
- Let $(\ ,\ )$ be the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that all short roots $\alpha$ satisfy $(\alpha, \alpha) = 2$.
- Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.
- Let $\varpi_1, \dots, \varpi_N \in \mathfrak{h}^*$ be the fundamental weights.
- Let $\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z}\varpi_j$ and $\mathbf{Q} = \bigoplus_{j=1}^N \mathbb{Z}\alpha_j$ be the weight and root lattices, respectively.
- Let $\mathbf{P}^+ = \bigoplus_{j=1}^N \mathbb{N}_0\varpi_j$ be the dominant integral weights.
- Let $W$ be the Weyl group of $\mathfrak{g}$.

# Quantized universal enveloping algebras

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## Definition

The *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is the algebra with generators $E_j, F_j$ for $1 \leq j \leq N$ and $K_\lambda$ for $\lambda \in \mathbf{P}$ satisfying

$$K_0 = 1, \qquad K_\lambda K_\mu = K_{\lambda+\mu}$$

$$K_\lambda E_j K_\lambda^{-1} = q^{(\lambda,\alpha_j)} E_j, \qquad K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda,\alpha_j)} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0 \qquad i \neq j$$

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Here $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ denotes the $q$-binomial coefficient, with $[n]_q! = \frac{q^n - q^{-n}}{q - q^{-1}}$.

## Example: $U_q(\mathfrak{sl}(2,\mathbb{C}))$

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In the simplest case $G = SL(2,\mathbb{C})$ this can be phrased more directly.

## Definition

The quantized universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is the algebra with generators $E, F$ and $K$ satisfying

$$KEK^{-1} = qE, \qquad KFK^{-1} = q^{-1}F$$

$$[E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

The generators should be thought of as deformations of the standard generators

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}(2,\mathbb{C})$. More precisely, $K$ corresponds to $q^{\frac{1}{2}H}$.

# Representation theory and representative functions

The finite dimensional representation theory of $U_q(\mathfrak{g})$ is similar to the one for the classical universal enveloping algebra $U(\mathfrak{g})$.

In particular, for every $\mu \in \mathbf{P}^+$ there exists a unique irreducible representation $V(\mu)$ with a highest weight vector $v_\mu$ satisfying

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## Definition

The algebra $\mathcal{O}(K_q) \subset U_q(\mathfrak{g})^*$ of representative functions on the quantum group $K_q$ is the algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbf{P}^+$.

This is a deformation of the algebra $\mathcal{O}(K)$ of polynomial functions on $K$.

There is a canonical bilinear pairing $(\ ,\ ) : U_q(\mathfrak{g}) \times \mathcal{O}(K_q) \to \mathbb{C}$ given by evaluation.

# Example: $SU_q(2)$

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Consider again the case $G = SL(2, \mathbb{C})$.

The algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$ has a unique irreducible weight representation $V(n)$ for every $n \in \frac{1}{2}\mathbb{N}_0$.

The $*$-algebra of representative functions $\mathcal{O}(SU_q(2))$ can be identified with the universal $*$-algebra generated by elements $\alpha$ and $\gamma$ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary, and give precisely Woronowicz's definition of quantum $SU(2)$.

For $q = 1$ this reproduces the algebra $\mathcal{O}(SU(2))$ of polynomial functions on the classical group $SU(2)$.

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**Fact**

All $\mathcal{D}(K_q)$ are isomorphic! In particular $\mathcal{D}(K_1)$ is noncommutative, and looks very different from functions on the group $AN$.

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Consider the vector space

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equipped with the multiplication

$$(x \bowtie f)(y \bowtie g) = x(f_{(1)}, y_{(1)})y_{(2)} \bowtie f_{(2)}(f_{(3)}, \hat{S}(y_{(3)}))g$$

and $*$-structure

$$(x \bowtie f)^* = (1 \bowtie f^*)(x^* \bowtie 1).$$

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### Definition

The full group $C^*$-algebra $C^*_{\mathrm{f}}(G_q)$ of the complex quantum group $G_q$ is the enveloping $C^*$-algebra of $\mathcal{D}(G_q)$.

## The representation theory of $G_q$

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By construction, a nondegenerate representation of $C^*_{\mathrm{f}}(G_q)$ on a Hilbert space $\mathcal{H}$ is the same thing as a nondegenerate $*$-homomorphism $\mathcal{D}(G_q) \to B(\mathcal{H})$.

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This is also the same thing as a unitary *Yetter-Drinfeld module*, that is, a pair of

- a unital $*$-representation $\alpha : \mathcal{O}(K_q) \to B(\mathcal{H})$,
- a nondegenerate $*$-representation $\gamma : \mathcal{D}(K_q) \to B(\mathcal{H})$,

satisfying the *Yetter-Drinfeld compatibility condition*

$$(f_{(1)}, x_{(1)})\alpha(f_{(2)})(\gamma(x_{(2)})(\xi)) = (f_{(2)}, x_{(2)})\gamma(x_{(1)})(\alpha(f_{(1)})(\xi))$$

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**Task**

Classify all irreducible Yetter-Drinfeld modules!

# Principal series representations

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For $\mu \in \mathbf{P}$ let

$$\mathcal{O}(\mathcal{E}_\mu) = \{f \in \mathcal{O}(K_q) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes 1\} \subset \mathcal{O}(K_q),$$

where $\pi : \mathcal{O}(K_q) \to \mathcal{O}(T)$ is the restriction of "functions" to the classical maximal torus.

For $q = 1$ this can be viewed as the space of (algebraic) sections of the homogeneous vector bundle $K \times_T \mathbb{C}_\mu$ associated to $\mu$.

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For any $\lambda \in \mathfrak{h}^*$ the space $\mathcal{O}(\mathcal{E}_\mu)$ becomes a Yetter-Drinfeld module via

$$f \cdot \xi = f_{(1)}\, \xi\, S(f_{(3)})(K_{\lambda+\rho}, S(f_{(2)}))$$
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$$\begin{aligned} f \cdot \xi &= f_{(1)}\, \xi\, S(f_{(3)})(K_{\lambda+\rho}, S(f_{(2)})) \\ x \cdot \xi &= (x, \xi_{(1)})\xi_{(2)} \end{aligned}$$

for $f \in \mathcal{O}(K_q), x \in \mathcal{D}(K_q)$.

## Definition

This is the *principal series Yetter-Drinfeld module*, or principal series representation, with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}^*$, and denoted $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$.

If $\lambda \in \mathfrak{t}^* \subset \mathfrak{h}^*$ then $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$ is a unitary Yetter-Drinfeld module with respect to the inner product on $\mathcal{O}(\mathcal{E}_\mu)$ induced by the Haar state of $\mathcal{O}(K_q)$.

# The structure of principal series representations

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One has $\mathcal{O}(\mathcal{E}_{\mu,\lambda}) = \mathcal{O}(\mathcal{E}_{\mu,\lambda'})$ as Yetter-Drinfeld modules if $\lambda - \lambda' \in i\hbar^{-1}\mathbf{Q}^\vee$, where $\hbar = \frac{h}{2\pi}$ and $\mathbf{Q}^\vee$ is the coroot lattice.

Therefore it is natural to consider parameters $\lambda$ in the torus

$$\mathfrak{t}_q^* = \mathfrak{t}^*/i\hbar^{-1}\mathbf{Q}^\vee$$

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**Theorem**

*Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}_q^*$. Then the principal series modules with parameters $(\mu, \lambda)$ and $(\mu', \lambda')$ are equivalent iff $(\mu', \lambda') = (w \cdot \mu, w \cdot \lambda)$ for some element $w$ in the Weyl group of $\mathfrak{g}$.*

These results are essentially due to Joseph-Letzter and depend on deep facts about the structure of $U_q(\mathfrak{g})$.

# The Plancherel formula

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## Theorem (V.-Yuncken 2019)

*Let $G_q$ be a complex quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu,\nu})_{\mu,\nu}$ be the Hilbert space bundle of unitary principal series representations over $\mathbf{P} \times \mathfrak{t}_q^*$. Then there is a unitary isomorphism*

$$Q : L^2(G_q) \cong \bigoplus_{\mu \in \mathbf{P}} \int_{\nu \in \mathfrak{t}_q^*}^{\oplus} HS(\mathcal{H}_{\mu,\nu}) dm_\mu(\nu)$$

*for the measures $dm_\mu$ on $\mathfrak{t}_q^*$ given by*

$$dm_\mu(\nu) = \prod_{\alpha \in \Delta^+} (q_\alpha^{1/2} - q_\alpha^{-1/2})^2 [(\mu + \nu)_\alpha]_{q_\alpha^{1/2}} [(\mu - \nu)_\alpha]_{q_\alpha^{1/2}} \, d\nu,$$

*where $d\nu$ denotes normalized Lebesgue measure on $\mathfrak{t}_q^*$.*

One also has to specify Duflo-Moore operators $D_{\mu,\nu}$ since the dual Haar weight of $G_q$ fails to be tracial.

In the simplest case $G_q = SL_q(2, \mathbb{C})$ the above result was obtained by Buffenoir-Roche (1999).

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This is proved by verifying the Plancherel formula

$$\epsilon_{G_q}(f) = \sum_{\mu \in \mathbf{P}} \int_{\mathfrak{t}_q^*} \mathrm{tr}(\pi_{\mu,\nu}(f) D_{\mu,\nu}^{-2}) dm_\mu(\nu)$$

for elements of the form $f = u_{ij}^\beta \otimes \omega_{kl}^\gamma \in \mathcal{O}(K_q) \otimes \mathcal{D}(K_q)$.

## Key fact

The right hand side identifies with Lefschetz numbers of certain Bernstein-Gelfand-Gelfand-complexes of Harish-Chandra modules.

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### Key fact

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### Remark

*The lowest order contribution in h of the quantum Plancherel measure agrees with the classical Plancherel measure*

$$\prod_{\alpha \in \Delta^+} |(\mu_\alpha + \nu_\alpha)|^2 d\nu = (\mu + \nu)_\alpha (\mu - \nu)_\alpha d\nu$$

*on* $\mathbf{P} \times \mathfrak{t}^*$.

# The reduced dual of $G_q$

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**Theorem (V.-Yuncken 2019)**

*Let $G_q$ be a complex quantum group and let $\mathcal{H} = (\mathcal{H}_{\mu,\nu})_{\mu,\nu}$ be the Hilbert space bundle of principal series representations of $G_q$ over $\mathbf{P} \times \mathfrak{t}_q^*$. Then the canonical $*$-homomorphism*

$$\pi : C_r^*(G_q) \to C_0(\mathbf{P} \times \mathfrak{t}_q^*, \mathbb{K}(\mathcal{H}))^W$$

*is an isomorphism.*

This follows by combining the Plancherel theorem with Dixmier's Stone-Weierstrass theorem.

The above result illustrates the fact that $G_q$ can indeed be viewed as a deformation of $G$, obtained by scaling the torus $\mathfrak{t}_q^*$ to infinite radius.

# The Connes-Kasparov conjecture

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Let $\mathfrak{k}$ be the Lie algebra of $K$, and let $\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$ be its dual space.

The group $K$ acts linearly on $\mathfrak{k}^*$ via the coadjoint action, and the corresponding semidirect product group

$$G_c = K \ltimes \mathfrak{k}^*$$

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Connes-Higson constructed a continuous field of groups $\mathbf{G}$ over $[0, 1]$ such that $\mathbf{G}_t = G$ for $t > 0$ and $\mathbf{G}_0 = G_c$.

**Theorem (Connes-Kasparov conjecture)**

*The continuous field $C^*_r(\mathbf{G})$ is trivial away from 0, and the induced map*

$$\mu : K_*(C^*_r(G_c)) = K_*(K \ltimes_{ad} C_0(\mathfrak{k})) \to K_*(C^*_r(G))$$

*is an isomorphism.*

This is a special case of the Baum-Connes conjecture - it holds by work Penington-Plymen.

# The Connes-Kasparov conjecture in the quantum case

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Let $G_1 = K \bowtie \hat{K}$ be the Drinfeld double of the *classical* group $K$. Then

$$C^*_{\mathrm{f}}(G_1) = C^*_{\mathrm{r}}(G_1) = K \ltimes_{ad} C(K).$$

Let us call this the *quantum Cartan motion group*.

Note that the quantum Cartan motion group is the "complex quantum group" $G_q = K_q \bowtie \hat{K}_q$ at $q = 1$.

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Recall that our deformation parameter is $q = e^h$.

Varying $t \in [0,1]$ determines a continuous field of $C^*$-algebras with fibers $C^*_{\mathrm{r}}(G_{e^{th}})$, the *quantum assembly field*.

**Theorem (Monk-V. 2018)**

*The quantum assembly field is trivial away from* 0 *and induces an isomorphism*

$$\mu_q : K_*(K \ltimes_{ad} C(K)) \to K_*(C^*_{\mathrm{r}}(G_q)).$$

# The Connes-Kasparov conjecture in the quantum case

## The Connes-Kasparov conjecture in the quantum case

In fact, one obtains a commutative diagram

$$\begin{array}{ccc} K_*(K \ltimes_{ad} C(K)) & \xrightarrow{\mu_q} & K_*(C^*_{\mathrm{r}}(G_q)) \\ \uparrow & & \uparrow \\ K_*(K \ltimes_{ad} C_0(\mathfrak{k})) & \xrightarrow{\mu} & K_*(C^*_{\mathrm{r}}(G)) \end{array}$$

The horizontal maps are bijective, and the vertical maps are split injective.

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This shows in particular that the classical assembly map is a direct summand of the quantum assembly map.

# Quantum Connes-Kasparov in the Meyer-Nest picture

# Quantum Connes-Kasparov in the Meyer-Nest picture

The construction of $\mu_q$ has an abstract reformulation in the spirit of Meyer-Nest's approach to the Baum-Connes conjecture.

The basic ingredient is the Kasparov category $KK^{G_q}$ associated to the quantum group $G_q$, and two complementary categories

$$\langle \mathcal{CI} \rangle = \langle \mathrm{ind}_{K_q}^{G_q}(A) \mid A \in KK^{K_q} \rangle$$
$$\mathcal{CC} = \{ B \mid \mathrm{res}_{K_q}^{G_q}(B) \cong 0 \text{ in } KK^{K_q} \}$$

of compactly induced and compactly contractible actions, respectively.

Using general machinery from triangulated categories, one obtains a simplicial approximation $D : \mathcal{P}_q \to \mathbb{C}$ of the trivial action, with $\mathcal{P}_q \in \langle \mathcal{CI} \rangle$ and $cone(D) \in \mathcal{CC}$.

## Theorem (V. 2019)

*The quantum assembly map $\mu_q$ identifies canonically with the induced map*

$$K_*(G_q \ltimes_{\mathrm{r}} \mathcal{P}_q) \to K_*(G_q \ltimes_{\mathrm{r}} \mathbb{C}) = K_*(C^*_{\mathrm{r}}(G_q))$$

This approach allows one to define an assembly map with arbitrary coefficients.