Representations and Hecke algebras for p-adic classical groups

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Notation

F a locally compact non-archimedean local field,

$$\mathfrak{o}_F, \ \ \mathfrak{p}_F, \ \ k_F = \mathfrak{o}_F/\mathfrak{p}_F, \ \ p = \operatorname{char} k_F,$$

For G the points of a connected reductive group over F,

- $\operatorname{Rep}(G)$ the category of smooth complex representations of G,

$$\pi : G \to \operatorname{End}_{\mathbb{C}}(V_\pi) \quad \text{such that} \quad V_\pi = \bigcup_{K \text{ open}} V_\pi^K$$

- $\operatorname{Irr}(G)$ the irreducible complex representations of G.

Cuspidal representations

For $P = MN$ a parabolic subgroup of G, we have a parabolic induction functor

$$\mathrm{Ind}_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G).$$

Then $\pi \in \mathrm{Rep}(G)$ is cuspidal if

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) = 0$$

for all *proper* parabolics $P = MN$ and $\sigma \in \mathrm{Rep}(M)$.

Cuspidal support

[Harish-Chandra]

Given $\pi \in \mathrm{Irr}(G)$ there are

- M a Levi subgroup, and
- σ an irreducible cuspidal representation of M,

such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$.

This cuspidal pair (M, σ) is unique up to conjugacy, and is called the cuspidal support of π.

Bernstein decomposition

A character $\chi : G \to \mathbb{C}^\times$ is unramified if it is trivial on every compact subgroup.

Cuspidal pairs (M, σ), (M', σ') are inertially equivalent if there is an unramified character χ of M such that

$(M, \sigma \otimes \chi)$, (M', σ') are conjugate in G.

[Bernstein]
For $\mathfrak{B}(G)$ the set of inertial equivalence classes $[M, \sigma]_G$,

$$\mathrm{Rep}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}_{\mathfrak{s}}(G),$$

where $\pi \in \mathrm{Rep}_{\mathfrak{s}}(G)$ iff all its irreducible subquotients have cuspidal support in $\mathfrak{s}$.

Bushnell–Kutzko's Theory of Types

Goal: for each inertial class $\mathfrak{s}$, find a type: a pair (J, λ) with

- J a compact open subgroup, and
- λ an irreducible representation of J,

such that, for $\pi \in \mathrm{Irr}(G)$,

$$\begin{aligned} \pi \in \mathrm{Irr}_{\mathfrak{s}}(G) &\iff \mathrm{Hom}_J(\lambda, \pi) \neq 0 \\ &\iff \mathrm{Hom}_G(\text{c-Ind}_J^G \lambda, \pi) \neq 0. \end{aligned}$$

Then $\text{c-Ind}_J^G \lambda$ is a progenerator for $\mathrm{Rep}_{\mathfrak{s}}(G)$.

More generally, do this for a finite set of inertial classes.

Cuspidal Types

For $\mathfrak{s} = [G, \pi]_G$ (i.e. π a cuspidal of G) this is roughly the same as constructing

- $\mathbf{J}$ a compact-mod-centre open subgroup, and
- $\boldsymbol{\lambda}$ an irreducible representation of $\mathbf{J}$,

such that

$$\pi \simeq \text{c-Ind}_{\mathbf{J}}^{G} \boldsymbol{\lambda}.$$

[Take $J \subseteq \mathbf{J}$ maximal compact and $\lambda \hookrightarrow \boldsymbol{\lambda}|_J$.]

[Bushnell–Kutzko]: $\mathrm{GL}_n(F)$ and $\mathsf{SL}_n(F)$.

[Yu, Fintzen]: arbitrary G when $p >$ order of Weyl group.

[Sécherre–S.]: inner forms of $\mathrm{GL}_n(F)$.

[S., Skodlerack]: classical groups, and inner forms, when $p \neq 2$.

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Non-cuspidal Types

For $\mathfrak{s} = [M, \sigma]_G$ there is a technique to build, from

- (J_M, λ_M) a type for $\sigma \in \mathrm{Cusp}(M)$,

a type in G (called a cover) such that

- ∘ $J = (J \cap \overline{N})(J \cap M)(J \cap N)$ and $J \cap M = J_M$, and
- ∘ $\lambda|_{J_M} \simeq \lambda_M$ and λ is trivial on $J \cap \overline{N}$, $J \cap N$,

where $P = MN$ with opposite $\overline{P} = M\overline{N}$.

[Bushnell–Kutzko]: $\mathrm{GL}_n(F)$.

[Goldberg–Roche]: $\mathsf{SL}_n(F)$.

[Kim–Yu, Fintzen]: arbitrary G when $p >$ order of Weyl group.

[Sécherre–S.]: inner forms of $\mathrm{GL}_n(F)$.

[Miyauchi–S.]: classical groups when $p \neq 2$.

Coarse structure of types (J, λ)

The group J has pro-p subgroups $J_{0+} \supseteq H_{0+}$ such that

J/J_{0+} is a finite reductive group

and λ decomposes as $\lambda = \kappa \otimes \sigma$, where

- κ is a representation of J with
  - $\kappa|_{J_{0+}}$ irreducible, and
  - $\kappa|_{H_{0+}}$ a multiple of a character θ,
- σ is a representation of J/J_{0+}.

One extreme case is when

- J is a parahoric subgroup,
- $J_{0+} = H_{0+}$ is its pro-p-radical,
- κ and θ are trivial.

Parahoric subgroups

Parahoric subgroups are certain compact open subgroups G_x of G, with

- a filtration $G_{x,r}$ indexed by $r \in \mathbb{R}_{\geq 0}$,
- $G_{x,r+} := \bigcup_{s>r} G_{x,s}$, a pro-$p$-group for $r \geq 0$, and
- $G_{x,0}/G_{x,0+}$ a (finite) connected reductive group over k_F.

In $\mathrm{GL}_n(F)$, we can have

- $G_{x,0} = \mathrm{GL}_n(\mathfrak{o}_F), \quad G_{x,r} = 1 + \mathbb{M}_n(\mathfrak{p}_F^{\lceil r \rceil})$,
- $G_{y,0} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}^\times, \quad G_{y,r} = \begin{cases} 1 + \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}, & 0 < r \leq \frac{1}{2}, \\ 1 + \begin{pmatrix} \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F \end{pmatrix}, & \frac{1}{2} < r \leq 1. \end{cases}$

Depth decomposition

For $\pi \in \mathrm{Irr}(G)$, its normalized depth is

$$d(\pi) = \inf\{r \geq 0 : \pi^{G_{x,r+}} \neq 0 \text{ for some parahoric } G_x\}.$$

[Moy–Prasad]: This lies in $\frac{1}{e_G}\mathbb{Z}$, for some integer e_G.

[Vignéras]:

$$\mathrm{Rep}(G) = \prod_{r \geq 0} \mathrm{Rep}_r(G),$$

where $\pi \in \mathrm{Rep}_r(G)$ iff all its irreducible subquotients have depth r.

$$\bigoplus_{G_{x,0} \text{ maximal}/\sim} \text{c-Ind}_{G_{x,0+}}^{G} \mathbf{1} \text{ is a pro-generator for } \mathrm{Rep}_0(G).$$

General linear groups

The characters θ which arise in the types are called semisimple characters. Every irreducible representation of G contains a semisimple character [Dat].

[Bushnell–Henniart, Kurinczuk–Skodlerack–S.]:
Intertwining is an equivalence relation on semisimple characters.

An intertwining class is called an endo-parameter and we write $\mathcal{EP}(G)$ for the set of endo-parameters for G.

Category decomposition by endo-parameter

$$\mathrm{Rep}(G) = \prod_{\mathfrak{t} \in \mathcal{EP}(G)} \mathrm{Rep}^{\mathfrak{t}}(G).$$

For suitable θ with endo-parameter $\mathfrak{t}$, the representation $\text{c-Ind}_{H_{0+}}^{G} \theta$ is a progenerator of $\mathrm{Rep}^{\mathfrak{t}}(G)$.

Classical groups ($p \neq 2$)

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Category decomposition by endo-parameter

$$\mathrm{Rep}(G) = \prod_{\mathfrak{t}\in\mathcal{EP}(G)} \mathrm{Rep}^{\mathfrak{t}}(G).$$

There is a finite set of $\theta^{(i)}$ with endo-parameter $\mathfrak{t}$, such that $\bigoplus_i \text{c-Ind}_{H_{0+}^{(i)}}^{G}\,\theta^{(i)}$ is a progenerator of $\mathrm{Rep}^{\mathfrak{t}}(G)$.

Change of coefficients

Suppose now R is an algebraically closed field of characteristic $\ell \neq p$.

Major change: $\pi \in \mathrm{Irr}_R(G)$ is

- cuspidal if π is not a subrepresentation of any $\mathrm{Ind}_P^G \sigma$;
- supercuspidal if π is not a subquotient of any $\mathrm{Ind}_P^G \sigma$.

These are not equivalent in general (unlike for $R = \mathbb{C}$).

For $\pi \in \mathrm{Irr}_R(G)$ there is a supercuspidal pair (M, σ) such that π is a subquotient of $\mathrm{Ind}_P^G \sigma$.

(M, σ) is not unique up to conjugacy in general!

General linear groups and inner forms

[Vignéras, Sécherre–S.]: For $\mathfrak{B}(G)$ the set of inertial equivalence classes of supercuspidal pairs,

$$\mathrm{Rep}_R(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Rep}_R^{\mathfrak{s}}(G),$$

where $\pi \in \mathrm{Rep}_R^{\mathfrak{s}}(G)$ iff all its irreducible subquotients have supercuspidal support in $\mathfrak{s}$.

Category decomposition by endo-parameter

$$\mathrm{Rep}_R(G) = \prod_{\mathfrak{t}\in\mathcal{EP}(G)} \mathrm{Rep}_R^{\mathfrak{t}}(G).$$

For suitable θ with endo-parameter $\mathfrak{t}$, the representation $\text{c-Ind}_{H_{0+}}^{G}\,\theta$ is a progenerator of $\mathrm{Rep}_R^{\mathfrak{t}}(G)$.

The latter works for R any commutative $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra.

Classical groups ($p \neq 2$)

We no longer have a decomposition by supercuspidal support but we still have:

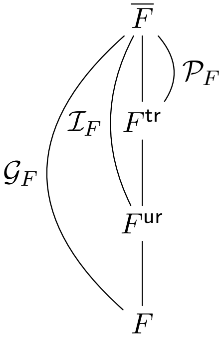
Category decomposition by endo-parameter

$$\mathrm{Rep}_R(G) = \prod_{\mathfrak{t} \in \mathcal{EP}(G)} \mathrm{Rep}_R^{\mathfrak{t}}(G).$$

There is a finite set of $\theta^{(i)}$ with endo-parameter $\mathfrak{t}$, such that $\bigoplus_i \text{c-Ind}_{H_{0+}^{(i)}}^G \theta^{(i)}$ is a progenerator of $\mathrm{Rep}_R^{\mathfrak{t}}(G)$.

This works for R any commutative $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra.

Comparison with Langlands parameters



maximal tame extension

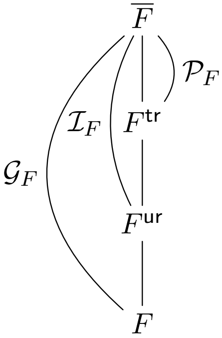
maximal unramified extension

$\mathcal{G}_F/\mathcal{I}_F \simeq \mathrm{Gal}(\overline{k_F}/k_F) \simeq \widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$.

$\mathcal{I}_F/\mathcal{P}_F \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell$.

$\mathcal{P}_F$ is a pro-p group (wild inertia group).

Comparison with Langlands parameters



maximal tame extension

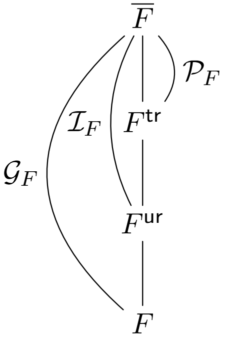
maximal unramified extension

$$\mathcal{G}_F/\mathcal{I}_F \simeq \mathrm{Gal}(\overline{k_F}/k_F) \simeq \widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell.$$

The Weil group $\mathcal{W}_F$ has

$$\mathcal{W}_F/\mathcal{I}_F \simeq \mathbb{Z} = \langle \mathrm{Frob} \rangle \hookrightarrow \widehat{\mathbb{Z}} \simeq \mathcal{G}_F/\mathcal{I}_F.$$

Comparison with Langlands parameters



maximal tame extension

maximal unramified extension

The Weil group has a filtration $\mathcal{W}_F^r$, with

- $\mathcal{W}_F^0 = \mathcal{I}_F$,
- $\mathcal{W}_F^{0+} = \mathcal{P}_F$.

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Depth of Langlands parameters

The Weil group has a filtration $\mathcal{W}_F^r$, with

- $\mathcal{W}_F^0 = \mathcal{I}_F$,
- $\mathcal{W}_F^{0+} = \mathcal{P}_F$.

The depth of $\phi_\pi : \mathcal{W}_F \times \mathsf{SL}_2(\mathbb{C}) \to \widehat{G} \rtimes \mathcal{W}_F$ a Langlands parameter for G is

$$d(\phi) := \inf\{r \geq 0 : \phi|_{\mathcal{W}_F^{r+}} \text{ is trivial}\}.$$

If $\pi \in \mathrm{Irr}(G)$ and ϕ_π is the corresponding Langlands parameter then in many (but not all) cases

$$d(\pi) = d(\phi_\pi).$$

(True for $\mathrm{GL}_n(F)$ and when p is large enough.)

General linear groups

Recall $\mathcal{EP}(G)$ is the set of endo-parameters for G.

Ramification Theorem [Bushnell–Henniart]

$$\begin{array}{ccc}
\mathrm{Irr}(G) & \xrightarrow{\sim} & \{\text{Langlands parameters for } G\} \\
\downarrow & & \downarrow \\
\mathcal{EP}(G) & \xrightarrow{\sim} & \{n\text{-dimensional reps of } \mathcal{P}_F\}^{\mathcal{W}_F}
\end{array}$$

Classical groups ($p \neq 2$)

Recall $\mathcal{EP}(G)$ is the set of endo-parameters for G.

Expectation/Question

$$\begin{array}{ccc}
\mathrm{Irr}(G) & \xrightarrow{\sim} & \{\text{enhanced Langlands parameters for } G\} \\
\downarrow & & \downarrow \\
\mathcal{EP}(G) & \xrightarrow{\sim} & \{\text{enhanced wild parameters for } G\} \\
\downarrow & & \downarrow \\
\mathcal{EP}(\mathrm{GL}_{n_G}) & \xrightarrow{\sim} & \{n_G\text{-dimensional reps of } \mathcal{P}_F\}^{\mathcal{W}_F}
\end{array}$$

Question

- Is there a theory of endo-parameters for an arbitrary group G which:
  - reflects restriction to wild inertia on the Galois side,
  - behaves functorially in the group,
  - gives a decomposition of $\mathrm{Rep}_R(G)$ for any commutative $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra R?

Further decomposition of the category: depth zero

[Dat, Lanard]

Using different methods (systems of idempotents), decompose the depth-zero subcategory

$$\mathrm{Rep}_R^0(G)$$

further:

- When $R = \overline{\mathbb{Z}}[\frac{1}{p}]$, and G is quasi-split and tamely ramified, $\mathrm{Rep}_R^0(G)$ is indecomposable.
- When $R = \overline{\mathbb{Z}}_\ell$ and G is split over an unramified extension, get a fine decomposition.
- When $R = \overline{\mathbb{Z}}_\ell$ and G is simply connected, describe the block decomposition of the subcategory of unipotent representations.

Structure of blocks (complex representations)

If $\mathrm{Rep}_{\mathfrak{s}}(G)$ is a Bernstein block and $P_{\mathfrak{s}}$ is a progenerator then we have

$$\mathrm{Rep}_{\mathfrak{s}}(G) \xleftrightarrow{\sim} \mathrm{End}_G(P_{\mathfrak{s}})\text{-Mod}$$

$$V \mapsto \mathrm{Hom}_G(P_{\mathfrak{s}}, V).$$

Possible questions:

1. Identify $\mathrm{End}_G(P_{\mathfrak{s}})$ explicitly and understand its module category.
2. Find equivalence between $\mathrm{End}_G(P_{\mathfrak{s}})$ and a similar algebra for a "simpler" block (of a different group).

Structure of blocks (complex representations)

[Iwahori–Matsumoto, Borel]: take G split, $\mathfrak{s} = [T, \mathbf{1}]_G$ and $P_\mathfrak{s} = \text{c-Ind}_I^G \mathbf{1}$, with I an Iwahori subgroup: get Iwahori–Hecke algebra.

[Bushnell–Kutzko]: for $G = \mathrm{GL}_n(F)$, and $\mathfrak{s}$ arbitrary, use the progenerator $P_\mathfrak{s} = \text{c-Ind}_J^G \lambda$, from the type to get

- $\mathrm{End}_G(P_\mathfrak{s})$ a tensor product of affine Iwahori–Hecke algebras of type A,
- a group $G_\mathfrak{s}$ such that $\mathrm{Rep}_\mathfrak{s}(G) \simeq \mathrm{Rep}_\mathbf{1}(G_\mathfrak{s})$ (principal block).

[Heiermann, . . . , Solleveld]: for $\mathfrak{s} = [M, \sigma]_G$, use the Bernstein progenerator $P_\mathfrak{s} = \mathrm{Ind}_P^G(\text{c-Ind}_{M^1}^M(\sigma|_{M^1}))$ to get

- explicit description of $\mathrm{End}_G(P_\mathfrak{s})$; or
- a related algebra $\mathcal{H}_\mathfrak{s}$ such that $\mathrm{Rep}_\mathfrak{s}^{\mathsf{fin}}(G) \overset{\sim}{\longleftrightarrow} \mathcal{H}_\mathfrak{s}\text{-mod}$.

Structure of blocks (mod-ℓ representations): inner forms of $\mathrm{GL}_n(F)$

We have a block decomposition by types (J, λ) but $\text{c-Ind}_J^G \lambda$ is no longer projective:

- still get a bijection of irreducibles,
- $\mathrm{End}_G(\text{c-Ind}_J^G \lambda)$ no longer sees the structure of the block.

[Chinello]: One can use the coarse structure $\lambda = \kappa \otimes \sigma$ of types with endo-parameter $\mathfrak{t}$ to get

$$\mathrm{End}_G(P_{\mathfrak{t}}) \xleftrightarrow{\sim} \mathrm{End}_{G_{\mathfrak{t}}}(P_{\mathbf{1}}),$$

where $\mathbf{1}$ is the depth zero endo-parameter for some other group $G_{\mathfrak{t}}$.

[Dat]: For $\mathrm{GL}_n(F)$, further reduce from a depth zero block to a unipotent block.

Questions

1. Can one make a similar reduction to a depth zero/unipotent block for any group?

2. Somehow describe the structure of the principal block.

Thank you.