Hochschild homology of reductive p-adic groups

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I. Hochschild homology

Hochschild homology for commutative algebras

Definition

A unital $\mathbb{C}$-algebra, then

$$HH_n(A) = \mathrm{Tor}_n^{A \otimes A^{op}}(A, A)$$

Hochschild–Kostant–Rosenberg theorem

V: nonsingular complex affine variety (or smooth manifold)
$\mathcal{O}(V)$: algebra of regular functions on V

$$HH_n(\mathcal{O}(V)) \cong \Omega^n_{\mathrm{alg}}(V)$$
$$HH_n(C^\infty(V)) \cong \Omega^n_{sm}(V)$$

Theorem (consequence of Serre–Swan)

When V is compact

$$K_*(C(V)) \cong K_*(C^\infty(V)) \cong H^*_{DR}(V)$$

however: $HH_*(C(V)) = HH_0(C(V)) = C(V)$

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Hochschild homology for noncommutative algebras

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A unital $\mathbb{C}$-algebra, then

$$HH_n(A) = \mathrm{Tor}_n^{A \otimes A^{op}}(A, A)$$

- $HH_0(A) = A/[A, A]$
- gives linear functions on the Grothendieck group of finite dimensional A-modules
- can help to classify irreducible A-modules

- $HH_n(A)$ depends only on the category of A-bimodules (Morita invariance)
- $HH_n(A)$ is a module over the centre $Z(A)$
- $HH_*(A)$ relates to the (periodic) cyclic homology of A

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Crossed product algebras

A: commutative unital algebra (some functions on $\mathrm{Irr}(A)$)
G: finite group acting on A
$A \rtimes G =$ vector space $A \otimes_{\mathbb{C}} \mathbb{C}[G]$ with multiplication $gag^{-1} = g(a)$

Clifford theory

- For $\sigma \in \mathrm{Irr}(A)$: stabilizer group G_σ
- For $\rho \in \mathrm{Irr}(G_\sigma)$: $\sigma \otimes \rho \in \mathrm{Irr}(A \rtimes G_\sigma)$
- $\pi(\sigma, \rho) := \mathrm{ind}_{A \rtimes G_\sigma}^{A \rtimes G} (\sigma \otimes \rho) \in \mathrm{Irr}(A \rtimes G)$
- $\pi(\sigma, \rho) \cong \pi(\sigma', \rho')$ if and only if $(\sigma', \rho') = (\sigma \circ \mathrm{Ad}(g)^{-1}, \rho \circ \mathrm{Ad}(g)^{-1})$ for some $g \in G$
- sets up bijection between $\mathrm{Irr}(A \rtimes G)$ and extended quotient

$$\mathrm{Irr}(A)//G = \{(\sigma, \rho) : \sigma \in \mathrm{Irr}(A), \rho \in \mathrm{Irr}(G_\sigma)\}/G$$

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Crossed product algebras (2)

V: nonsingular complex affine variety
$A = \mathcal{O}(V)$, so $\mathrm{Irr}(A) = V$
G: finite group acting on V by algebraic automorphisms

Theorem (Brylinski, Nistor)

There is a natural isomorphism of $\mathcal{O}(V)^G$-modules
$HH_n(\mathcal{O}(V) \rtimes G) \cong \big(\bigoplus_{g \in G} \Omega^n(V^g) \big)^G$

This can also be interpreted as n-forms on

$$\{(v, g) \in V \times G : g(v) = v\} / G$$

where G acts by $g'(v, g) = (g'v, g'g(g')^{-1})$

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Twisted crossed product algebras

2-cocycle $\natural : G \times G \to \mathbb{C}^\times$

twisted group algebra $\mathbb{C}[G,\natural]$, has basis $\{T_g : g \in G\}$ and $T_g T_{g'} = \natural(g,g') T_{gg'}$

Definition

The twisted crossed product $\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]$ is the algebra like $\mathcal{O}(V) \rtimes G$, only now with $\mathbb{C}[G,\natural]$ as subalgebra

Clifford theory

- for $v \in V, \rho \in \mathrm{Irr}(\mathbb{C}[G_v,\natural])$:
- $\pi(v,\rho) := \mathrm{ind}_{\mathcal{O}(V) \rtimes \mathbb{C}[G_v,\natural]}^{\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]} (\mathbb{C}_v \otimes \rho) \;\in\; \mathrm{Irr}(\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural])$
- gives bijection between $\mathrm{Irr}(\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural])$ and twisted extended quotient

$$(V//G)_\natural := \{(v,\rho) : v \in V, \rho \in \mathrm{Irr}(\mathbb{C}[G_v,\natural])\}/G$$

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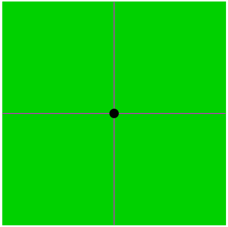
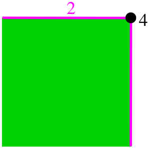
Example of a twisted extended quotient

$G = (\mathbb{Z}/2\mathbb{Z})^2$ acts on $[-1,1]^2 =$



(untwisted) extended quotient

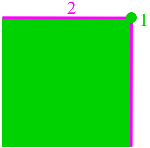
$$[-1,1]^2//G$$



$$\mathbb{C}[G] \cong \mathbb{C}^4$$

twisted extended quotient

$([-1,1]^2//G)_\natural$, $\natural$ nontrivial



$$\mathbb{C}[G,\natural] \cong M_2(\mathbb{C})$$

Twisted crossed product algebras (2)

For $g, h \in G$ define

$$\natural^g(h) = T_h T_g T_h^{-1} T_{hgh^{-1}}^{-1} \in \mathbb{C}^\times$$

- $\natural^g|_{Z_G(g)}$ is a character
- The $\natural^g$ measure how far away from a group algebra $\mathbb{C}[G, \natural]$ is:

$$|\mathrm{Irr}(\mathbb{C}[G, \natural])| = |\{g \in G : \natural^g|_{Z_G(g)} = 1\}/G\text{-conjugacy}|$$

Theorem

V nonsingular complex affine variety with action of finite group G
There is an isomorphism of $\mathcal{O}(V)^G$-modules
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When $\natural$ is nontrivial, $HH_n(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ consists of differential forms, but not (??) on an actual variety

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II. Some representation theory of reductive p-adic groups

Reductive p-adic groups

$\mathcal{G}$: connected reductive algebraic group
F: non-archimedean local field
$G = \mathcal{G}(F)$: reductive p-adic group, with locally compact topology

- $\mathrm{Rep}(G)$: category of smooth G-representations (on $\mathbb{C}$-vector spaces)
- $\mathcal{H}(G)$: convolution algebra of locally constant compactly supported functions $G \to \mathbb{C}$
- equivalence of categories $\mathrm{Rep}(G) \cong \mathrm{Mod}(\mathcal{H}(G))$

Goal: determine $HH_*(\mathcal{H}(G))$

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Bernstein decomposition

L: Levi subgroup of G

σ: irreducible supercuspidal G-representation

$X_{\mathrm{nr}}(L)$: group of unramified characters of L (a complex algebraic torus)

- $\mathfrak{s}$: equivalence class of (L, σ), up to tensoring with $X_{\mathrm{nr}}(L)$ and G-conjugation
- $\mathrm{Rep}(G)^{\mathfrak{s}}$: subcategory of $\mathrm{Rep}(G)$ generated by the parabolic inductions of the $\sigma \otimes \chi$ with $\chi \in X_{\mathrm{nr}}(L)$

Theorem (Bernstein)

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}$$

$$\mathcal{H}(G) = \bigoplus\nolimits_{\mathfrak{s}} \mathcal{H}(G)^{\mathfrak{s}}$$

where $\mathcal{H}(G)^{\mathfrak{s}}$ is the two sided ideal that annihilates all $\mathrm{Rep}(G)^{\mathfrak{s}'}$ with $\mathfrak{s}' \neq \mathfrak{s}$

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the Bernstein centre $Z(\mathrm{Rep}(G)^{\mathfrak{s}})$

- surjection $X_{\mathrm{nr}}(L) \to X_{\mathrm{nr}}(L)\sigma \subset \mathrm{Irr}(L)$
- finite fibers $X_{\mathrm{nr}}(L,\sigma) := \{\chi \in X_{\mathrm{nr}}(L) : \sigma \otimes \chi \cong \sigma\}$

- The group $N_G(L)/L$ acts naturally on $\mathrm{Irr}(L)$, by $(g\pi)(l) = \pi(g^{-1} l g)$
- for each $g \in N_G(L)/L$ which stabilizes $X_{\mathrm{nr}}(L)\sigma$: pick a lift to a transformation of $X_{\mathrm{nr}}(L)$
- $W(L,\mathfrak{s})$: group of transformations of $X_{\mathrm{nr}}(L)$ generated by $X_{\mathrm{nr}}(L,\sigma)$ and the above lifts

Theorem (Bernstein)

$Z(\mathrm{Rep}(G)^{\mathfrak{s}}) \cong \mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$
This is also $Z(\mathcal{H}_{\mathfrak{s}})$ if $\mathcal{H}_{\mathfrak{s}}$ is a unital algebra Morita equivalent with $\mathcal{H}(G)^{\mathfrak{s}}$

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This is also $Z(\mathcal{H}_{\mathfrak{s}})$ if $\mathcal{H}_{\mathfrak{s}}$ is a unital algebra Morita equivalent with $\mathcal{H}(G)^{\mathfrak{s}}$

the Bernstein centre $Z(\mathrm{Rep}(G)^{\mathfrak{s}})$

- surjection $X_{\mathrm{nr}}(L) \to X_{\mathrm{nr}}(L)\sigma \subset \mathrm{Irr}(L)$
- finite fibers $X_{\mathrm{nr}}(L,\sigma) := \{\chi \in X_{\mathrm{nr}}(L) : \sigma \otimes \chi \cong \sigma\}$
- The group $N_G(L)/L$ acts naturally on $\mathrm{Irr}(L)$, by $(g\pi)(l) = \pi(g^{-1}lg)$
- for each $g \in N_G(L)/L$ which stabilizes $X_{\mathrm{nr}}(L)\sigma$: pick a lift to a transformation of $X_{\mathrm{nr}}(L)$
- $W(L,\mathfrak{s})$: group of transformations of $X_{\mathrm{nr}}(L)$ generated by $X_{\mathrm{nr}}(L,\sigma)$ and the above lifts

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Reduction to simpler algebras

Bernstein constructed a finitely generated projective generator $\Pi_{\mathfrak{s}}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$

$\mathcal{H}_{\mathfrak{s}} := \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$ is some sort of Hecke algebra

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Equivalence of categories

$$\mathrm{Rep}(G)^{\mathfrak{s}} \to \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}}) : \pi \mapsto \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$$

Consequence of Morita invariance and additivity of HH_*:

There are isomorphisms of $\mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$-modules

$$HH_n(\mathcal{H}(G)) = \bigoplus_{\mathfrak{s}} HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \cong \bigoplus_{\mathfrak{s}} HH_n(\mathcal{H}_{\mathfrak{s}})$$

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Classification of irreducible representations

$\mathbb{C}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$: quotient field of $\mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})} = Z(\mathcal{H}_\mathfrak{s})$

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There exist a 2-cocycle $\natural_\mathfrak{s}$ of $W(L,\mathfrak{s})$ and an isomorphism of $\mathbb{C}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$-algebras

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Fix an isomorphism as above. There exists a canonical bijection

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Example

$G = SL_2(F), \mathfrak{s} = [T, \mathrm{triv}], X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times, W(T,\mathfrak{s}) = W(G,T)$

$\mathrm{Irr}(G)^\mathfrak{s}$	$\mathrm{Irr}(\mathcal{O}(\mathbb{C}^\times) \rtimes W)$	$R(\mathcal{O}(\mathbb{C}^\times) \rtimes W)$
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III. Hochschild homology of $\mathcal{H}(G)$

Hochschild homology for one Bernstein block

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$R(A)$: Grothendieck group of finite length A-modules

Theorem (variation on ABPS, reformulated)

An isomorphism as above induces a $\mathbb{Z}$-linear bijection

$$\zeta : R\big(\mathcal{O}(X_{\mathrm{nr}}(L)) \rtimes \mathbb{C}[W(L,\mathfrak{s}), \natural_\mathfrak{s}]\big) \to R(\mathcal{H}_\mathfrak{s})$$

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An isomorphism as above induces a $\mathbb{C}$-linear bijection

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Example: Iwahori-spherical Bernstein block for $SL_2(F)$

$L = T \cong F^\times, X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$, $W(L,\mathfrak{s}) = W(G,T) = \{1, w\}$, $\natural_{\mathfrak{s}} = 1$

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Harish-Chandra–Schwartz algebra of G

- $\mathcal{S}(G)$ is a topological algebra of locally constant functions $G \to \mathbb{C}$ that decay rapidly
- $\mathcal{S}(G)$ contains $\mathcal{H}(G)$ as a dense subalgebra
- $\mathrm{Rep}(\mathcal{S}(G))$ is equivalent with the category of tempered smooth G-representations

Decompositions of $\mathcal{S}(G)$

- Bernstein: $\mathcal{S}(G) = \bigoplus_{\mathfrak{s}} \mathcal{S}(G)^{\mathfrak{s}}$
- Harish-Chandra: For every $\mathfrak{s}$ there exists a finite set $\Delta_G^{\mathfrak{s}}$ of pairs $\mathfrak{d} = (M, \delta)$ with $M \subset G$ a Levi subgroup and δ in the discrete series of M, such that (in the same way as the Bernstein decomposition) $\mathcal{S}(G)^{\mathfrak{s}} = \bigoplus_{\mathfrak{d} \in \Delta_G^{\mathfrak{s}}} \mathcal{S}(G)^{\mathfrak{d}}$

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- Harish-Chandra: For every $\mathfrak{s}$ there exists a finite set $\Delta_G^{\mathfrak{s}}$ of pairs $\mathfrak{d} = (M, \delta)$ with $M \subset G$ a Levi subgroup and δ in the discrete series of M, such that (in the same way as the Bernstein decomposition) $\mathcal{S}(G)^{\mathfrak{s}} = \bigoplus_{\mathfrak{d} \in \Delta_G^{\mathfrak{s}}} \mathcal{S}(G)^{\mathfrak{d}}$

Example: $G = SL_2(F), \mathfrak{s} = [T, \mathrm{triv}]$

$\Delta_G^{\mathfrak{s}} = \{(G, \mathrm{St}), (T, \mathrm{triv})\}$

$\mathcal{S}(G)^{(G,\mathrm{St})}$ has St as only irrep

$\mathcal{S}(G)^{(T,\mathrm{triv})}$ corresponds to the unitary unramified principal series of G

Hochschild homology of $\mathcal{S}(G)$

- Hochschild homology of $\mathcal{S}(G)$ with respect to complete bornological tensor product
- for compact open subgroup K:
  $\mathcal{S}(G,K) =$ Fréchet algebra of K-biinvariant functions in $\mathcal{S}(G)$
- $\mathcal{S}(G) = \lim_K \mathcal{S}(G,K)$

Theorem (Brodzki–Plymen)

Continuity: $HH_n(\mathcal{S}(G)) = \lim_K HH_n(\mathcal{S}(G,K))$
Additivity: $HH_n(\mathcal{S}(G)) = \bigoplus_{\mathfrak{s}} HH_n(\mathcal{S}(G)^{\mathfrak{s}})$

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Hochschild homology of $\mathcal{S}(G)$ (2)

Unitary unramified characters

- $X_{\mathrm{unr}}(L)$: group of unitary unramified characters $L \to S^1 \subset \mathbb{C}^\times$
- $X_{\mathrm{nr}}(L)$ is a complex torus and $X_{\mathrm{unr}}(L)$ is the maximal compact real subtorus

Theorem

There is an isomorphism of Fréchet spaces

$$HH_n(\mathcal{S}(G)^{\mathfrak{s}}) \to HH_n\big(C^\infty(X_{\mathrm{unr}}(L)) \rtimes \mathbb{C}[W(L,\mathfrak{s}), \natural_{\mathfrak{s}}]\big)$$
$$\cong \Big(\bigoplus_{w \in W(L,\mathfrak{s})} \Omega^n_{sm}(X_{\mathrm{unr}}(L)^w) \otimes \natural_{\mathfrak{s}}^w \Big)^{W(L,\mathfrak{s})}$$

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$HH_n(\mathcal{H}(G)^{\mathfrak{s}})$ as module over the Bernstein centre

Theorem (reformulation of before)

Write $X_{\mathrm{nr}}(L)^w = \sqcup_{c \in \pi_0(X_{\mathrm{nr}}(L)^w)} X_{\mathrm{nr}}(L)^w_c$
There exists a $\mathbb{C}$-linear bijection

$$HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \to \Big(\bigoplus_{w \in W(L,\mathfrak{s})} \bigoplus_{c \in \pi_0(X_{\mathrm{nr}}(L)^w} \Omega^n(X_{\mathrm{nr}}(L)^w_c) \otimes \natural^w_{\mathfrak{s}} \Big)^{W(L,\mathfrak{s})}$$

- Each part $\Omega^n(X_{\mathrm{nr}}(L)^w_c)$ lands in $HH_n(\mathcal{S}(G)^{\mathfrak{d}})$ for a unique (up to equivalence) $\mathfrak{d} = (M, \delta)$ (with δ in the discrete series of M)
- $\mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$-character of $I^G_{MU}(\delta)$: $W(L,\mathfrak{s}) t_\delta$
- $|t_\delta| \in \mathrm{Hom}(L, \mathbb{R}_{>0})$

Proposition

In the above theorem, the action of $\mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}$ on the part of $HH_n(\mathcal{H}(G)^{\mathfrak{s}})$ from $\Omega^n(X_{\mathrm{nr}}(L)^w_c)$ is

$$(f\omega)(\chi) = f(|t_\delta|\chi)\omega(\chi) \quad f \in \mathcal{O}(X_{\mathrm{nr}}(L))^{W(L,\mathfrak{s})}, \omega \in \Omega^n(X_{\mathrm{nr}}(L)^w_c)$$

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IV. Other homology theories for reductive p-adic groups

Periodic cyclic homology

- Periodic cyclic homology is a noncommutative generalization of De Rham cohomology
- $HP_n(\mathcal{O}(V)) = H_{DR}^{\text{even/odd}}(V)$
- Connes constructed a differential B on $HH_*(A)$ such that (in good cases) $HP_n(A) = \bigoplus_{m\in\mathbb{Z}} H^{n+2m}(HH_*(A), B)$

Theorem

(i) Connes' differential B becomes the standard exterior differential on $HH_*(\mathcal{H}(G)^{\mathfrak{s}})$

(ii) $HP_n(\mathcal{H}(G)^{\mathfrak{s}}) \cong \bigoplus_{m\in\mathbb{Z}} \Big(\bigoplus_{w\in W(L,\mathfrak{s})} H_{dR}^{n+2m}(X_{\mathrm{nr}}(L)^w) \otimes \natural_{\mathfrak{s}}^w \Big)^{W(L,\mathfrak{s})}$

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Link with topological K-theory

Corollary

$\mathcal{H}(G)^{\mathfrak{s}} \to \mathcal{S}(G)^{\mathfrak{s}}$ induces an isomorphism on periodic cyclic homology

Theorem

The Chern character induces a $\mathbb{C}$-linear bijection

$$K_*(\mathcal{S}(G)^{\mathfrak{s}}) \otimes_{\mathbb{Z}} \mathbb{C} \to HP_*(\mathcal{S}(G)^{\mathfrak{s}})$$

Reduced C^*-algebra of G

- Bernstein decomposition: $C_r^*(G) = \bigoplus_{\mathfrak{s}} C_r^*(G)^{\mathfrak{s}}$
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Topological K-theory of reductive p-adic groups

Corollary

$$K_n(C_r^*(G)^{\mathfrak{s}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_n\big(C(X_{\mathrm{unr}}(L)) \rtimes \mathbb{C}[W(L,\mathfrak{s}),\natural]\big) \otimes_{\mathbb{Z}} \mathbb{C}$$
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- This is a twisted version of the $W(L,\mathfrak{s})$-equivariant K-theory of $X_{\mathrm{unr}}(L)$ (with complex coefficients)
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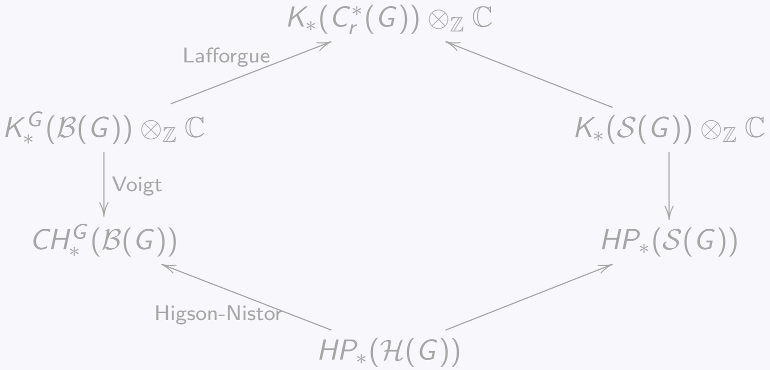
Relation with Baum–Connes conjecture

- $K_*^G(\mathcal{B}(G))$: equivariant K-homology of Bruhat–Tits building of G
- $CH_*^G(\mathcal{B}(G))$: equivariant chamber homology of $\mathcal{B}(G)$

Commutative diagram of isomorphisms



Open problem:

Describe the Bernstein decomposition on the left hand side

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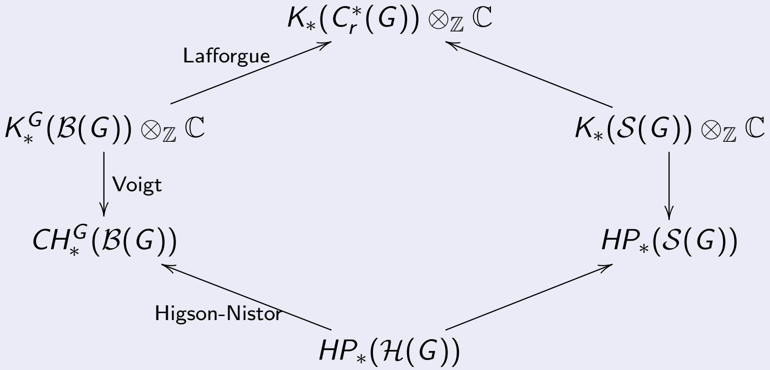

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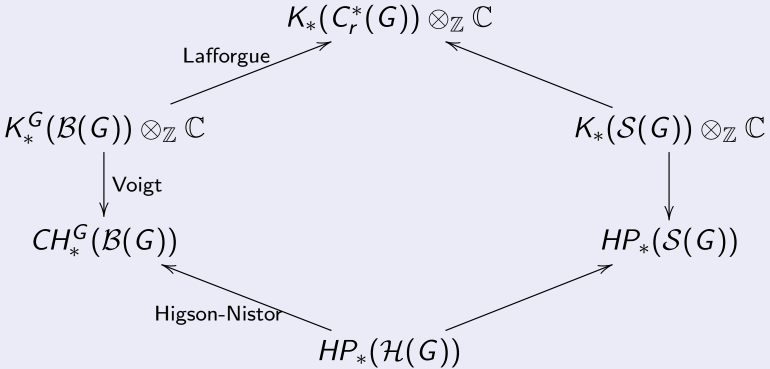

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