Towards an explicit Langlands correspondence of *p*-adic groups: the example of G_2

Anne-Marie Aubert

Institut de Mathématiques de Jussieu – Paris Rive Gauche
C.N.R.S., Sorbonne Université and Université de Paris Cité

LMS Bath Symposium
K-THEORY AND REPRESENTATION THEORY
18-22 July 2022

Extended quotients:

A tool to reduce LLC to supercuspidal cases

Definition

Let X be a space and Γ a finite group acting on X. For $x \in X$, let $\Gamma_x \subset \Gamma$ be the fixator of x: $\Gamma_x := \{\gamma \in \Gamma \,:\, \gamma \cdot x = x\}$.
The (spectral) extended quotient of X by Γ is the quotient

$$X/\!/\Gamma := \{(x, \tau) \,:\, x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\} /\Gamma.$$

Geometric extended quotient

Instead of irred. repres. of Γ_x, we can consider conjugacy classes in Γ_x, and form the quotient

$$\{(x, \tau) \,:\, x \in X, \gamma \in \Gamma_x\} /\Gamma.$$

An example of spectral extended quotient

Let T be a torus in a reductive group G and $W := \mathrm{N}_G(T)/T$ the corresponding Weyl group, acting on T by conjugation, then we can consider the extended quotient $T//W$.

Example that will be used in the talk (p-adic group side)

Let G be a p-adic reductive group, L a Levi subgroup of G and $\mathrm{Irr}_{\mathrm{cusp}}(L)$ the set of isomorphism classes of irreducible supercuspidal representations of L. The group $W(L) := \mathrm{N}_G(L)/L$ acts on $\mathrm{Irr}_{\mathrm{cusp}}(L)$ and we can form the panoramic (spectral) p-adic extended quotient:

$$\mathrm{Irr}_{\mathrm{cusp}}(L)//W(L).$$

Why "panoramic"?

Because we are considering the supercuspidal irreducible representations of L all together.

Example that will be used in the talk (Galois side)

Let $G^\vee$ and $L^\vee$ be the complex dual groups of G, L and $\Phi_{e,\mathrm{cusp}}(L)$ the set of $L^\vee$-conjugacy classes of cuspidal enhanced Langlands parameters for L. The group $W(L^\vee) := \mathrm{N}_{G^\vee}(L^\vee)/L^\vee$ acts on $\Phi_{e,\mathrm{cusp}}(L)$ and we can form the panoramic (spectral) Galois extended quotient:

$$\Phi_{e,\mathrm{cusp}}(L)//W(L^\vee).$$

Remark

The objects written in blue will be defined later in the talk!

Goals of the talk

Explain the meaning of the following conjecture, give examples for which it is satisfied, and suggest other examples to be studied:

Conjecture [A-Moussaoui-Solleveld]

The local Langlands correspondence induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)//W(L) \xleftrightarrow{1-1} \Phi_{e,\mathrm{cusp}}(L)//W(L^\vee),$$

for any Levi subgroup of G.

The cuspidality conjecture (special case of the above for $L = G$)

The local Langlands correspondence restricts to a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{e,\mathrm{cusp}}(G).$$

Notation (group side)

- F a non-archimedean local field
- $\mathbf{G}$ connected reductive algebraic group defined over F
- G the group of F-rational points of $\mathbf{G}$
- G^* the F-quasi-split inner form of G

Assumption [for simplicity of the exposition]

We suppose that G^* is F-split.

Notation (Galois side)

- W_F Weil group of F
- $G^\vee$ complex reductive group with root datum dual to that of G
- $\mathrm{Z}_{G^\vee}$ center of $G^\vee$
- $G^\vee_{\mathrm{der}}$ derived group of $G^\vee$
- $G^\vee_{\mathrm{sc}}$ simply connected cover of $G^\vee_{\mathrm{der}}$

Definition

A Langlands parameter – or L-parameter – is a morphism $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ such

- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is morphism of algebraic groups,
- $\varphi(w)$ is a semisimple element of $G^\vee$, for any $w \in W_F$.

Notation

Groups attached to an L-parameter φ:

- $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi) := \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi(W_F'))$, where $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$
- $\mathcal{S}_\varphi := \pi_0\left(\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)\right)$ component group of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)$

Definition

An enhanced L-parameter is a pair (φ, ρ) where φ is an L-parameter and ρ an irreducible representation of $\mathcal{S}_\varphi$.
The representation ρ is called an enhancement of φ.

Action of $G^\vee$ on the set of enhanced L-parameters:

$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho)$, where ${}^g\rho \colon h \mapsto \rho(g^{-1}hg)$ for $h \in \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)$ and $g \in G^\vee$.

Remark

The center of $\mathcal{S}_\varphi$ acts by a character on any $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, so any enhancement ρ of φ determines a character ζ_ρ of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}$.

The inner twists of G^* are parametrized by the Galois cohomology group $H^1(F, \mathbf{G}_{\mathrm{ad}})$, where $\mathbf{G}_{\mathrm{ad}}$ is the adjoint group of $\mathbf{G}$. The parametrization is canonically determined by requiring that G^* corresponds to the trivial element of $H^1(F, \mathbf{G}_{\mathrm{ad}})$. Kottwitz constructed a natural group homomorphism

$$H^1(F, \mathbf{G}_{\mathrm{ad}}) \xrightarrow{\sim} \mathrm{Irr}_{\mathbb{C}}(\mathrm{Z}_{G^\vee_{\mathrm{sc}}}). \quad (1)$$

Definition

An enhanced L-parameter (φ, ρ) is called G-relevant if ζ_ρ parametrizes G via (1). Notation:
$\Phi_{\mathrm{e}}(G) := \{G^\vee$-conjugacy classes of G-relevant enhanced L-parameters$\}$.

Remark

If (φ, ρ) is G-relevant, then φ is G-relevant in Borel's terminology.

LLC

The local Langlands correspondence is expected to provide a bijection

$$\mathrm{LLC}\colon \mathrm{Irr}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}(G),$$

where $\mathrm{Irr}(G)$ is the set of isomorphism classes of irreducible smooth representations of G.

Useful viewpoint

For φ a given L-parameter, we define

- $\mathcal{G}_\varphi := \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi(W_F))$ (a possibly disconnected complex reductive group); $\mathcal{G}_\varphi^\circ$ it identity component;
- $u_\varphi := \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) \in \mathcal{G}_\varphi^\circ$ (unipotent element);
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi))$.

We have

$$\mathcal{S}_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi).$$

Definition [A-Moussaoui-Solleveld (2015)]

An enhanced L-parameter $(\varphi, \varrho) \in \Phi_{\mathrm{e}}$ is called cuspidal if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$),
- (u_φ, ϱ) is a cuspidal pair in $\mathcal{G}_\phi$.

Notation: $\Phi_{\mathrm{e,cusp}} :=$ $G^\vee$-conjugacy of cuspidal enhanced L-parameters.

Cuspidal pairs in complex connected reductive groups

Let $\mathcal{G}$ be a possibly disconnected complex Lie group, with identity component $\mathcal{G}^\circ$. Let u be a unipotent element in $\mathcal{G}^\circ$, and let $A_{\mathcal{G}^\circ}(u)$ denote the group of components of $\mathrm{Z}_{\mathcal{G}^\circ}(u)$.
Let ϱ° be an irreducible representation of $A_{\mathcal{G}^\circ}(u)$. The pair (u, ϱ°) is called cuspidal if it determines a $\mathcal{G}^\circ$-equivariant cuspidal local system on the $\mathcal{G}^\circ$-conjugacy class of u as defined by Lusztig.

Remark

In particular, if (u, ϱ°) is cuspidal, then u is a distinguished unipotent element in $\mathcal{G}^\circ$ (that is, u does not meet the unipotent variety of any proper Levi subgroup of $\mathcal{G}^\circ$). However, in general not every distinguished unipotent element supports a cuspidal representation.

Generalized Springer variety

$\mathcal{P} = \mathfrak{L}\mathcal{U}$ parabolic subgroup of $\mathcal{G}^\circ$, let $u \in \mathcal{G}^\circ$ and $v \in \mathfrak{L}$ be unipotent elements. The group $\mathrm{Z}_{\mathcal{G}^\circ}(u) \times \mathrm{Z}_{\mathfrak{L}}(v)\mathcal{U}$ acts on the variety

$$Y_{u,v} := \left\{ y \in \mathcal{G}^\circ \;:\; y^{-1}uy \in v\mathcal{U} \right\}$$

by $(g, p) \cdot y := gyp^{-1}$, with $g \in \mathrm{Z}_{\mathcal{G}^\circ}(u)$, $p \in \mathrm{Z}_{\mathfrak{L}}(v)\mathcal{U}$ and $y \in Y_{u,v}$.

The group $A_{\mathcal{G}^\circ}(u) \times A_{\mathfrak{L}}(v)$ acts on the set of irreducible components of $Y_{u,v}$ of maximal dimension (i.e. $\dim \mathcal{U} + \frac{1}{2}(\dim \mathrm{Z}_{\mathcal{G}^\circ}(u) + \dim \mathrm{Z}_{\mathfrak{L}}(v))$). Let $\sigma_{u,v}$ denote the corresponding permutation representation.

Definition

Let $\varrho^\circ \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$. Then ϱ° is cuspidal if

$$\langle \varrho^\circ, \sigma_{u,v} \rangle_{A_{\mathcal{G}^\circ}(u)} \neq 0 \quad \Rightarrow \quad \mathcal{P} = \mathcal{G}^\circ,$$

where $\langle\ ,\ \rangle_{A_{\mathcal{G}^\circ}(u)}$ is the usual scalar product on the space of class functions on $A_{\mathcal{G}^\circ}(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

Cuspidal pairs in arbitrary complex reductive groups

The group $A_{\mathcal{G}^\circ}(u)$ may be viewed as a subgroup of the group $A_{\mathcal{G}}(u)$ of components of $\mathrm{Z}_{\mathcal{G}}(u)$. Let ϱ be an irreducible representation of $A_{\mathcal{G}}(u)$. We say that (u, ϱ) is a cuspidal pair if the restriction of ϱ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ϱ° such that one (or equivalently any) of the pairs (u, ϱ°) in $\mathcal{G}^\circ$ is cuspidal.

Remark: an equivalent definition of cuspidality

There is a parabolic induction functor

$$\mathrm{i}_{\mathfrak{L},\mathcal{P}}^{\mathcal{G}^\circ} \colon \mathrm{Perv}_{\mathfrak{L}}(\text{unipotent variety of } \mathfrak{L}) \longrightarrow \mathrm{Perv}_{\mathcal{G}^\circ}(\text{unipotent variety of } \mathcal{G}^\circ).$$

Then $\varrho^\circ \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$ is cuspidal iff $\mathrm{IC}(C_u, \mathcal{E}_{\varrho^\circ})$ does not occur in the image of $\mathrm{i}_{\mathfrak{L},\mathcal{P}}^{\mathcal{G}}$ for any proper parabolic subgroup $\mathcal{P}$. Here C_u is the $\mathcal{G}^\circ$-conjugacy class of u and $\mathcal{E}_{\varrho^\circ}$ the irreducible $\mathfrak{L}$-equivariant local system on C_u that corresponds to ϱ°.

Cuspidality Conjecture [A-Moussaoui-Solleveld]

The cuspidal G-relevant enhanced Langlands parameters correspond by the LLC to the irreducible supercuspidal representations of G:

$$\mathrm{LLC}\colon \mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(G). \tag{2}$$

State of art

The cuspidality conjecture is known to hold in all the cases where a LLC for supercuspidal representations has been constructed, i.e. in the following cases:

- general linear groups and split classical p-adic groups [Moussaoui, 2017] (based of works of Arthur and Moeglin),
- inner forms of linear groups and of special linear groups, and quasi-split unitary p-adic groups [A-Moussaoui-Solleveld, 2018],
- unipotent supercuspidal representations of an arbitrary group G ([Lusztig, 1995] + [Feng-Opdam-Solleveld, 2020]),
- non-singular ($\supset$ regular) supercuspidal representations of any group G such that $\mathbf{G}$ splits over tamely ramified extension of F and p is odd (follows from [Kaletha, 2019]).

Notation

- L Levi subgroup of a parabolic subgroup of G
- $W_G(L) := \mathrm{N}_G(L)/L$
- $W_{G^\vee}(L^\vee) := \mathrm{N}_{G^\vee}(L^\vee)/L^\vee$

Remark

There is a canonical bijection $w \mapsto w^\vee$ from $W_G(L)$ to $W_{G^\vee}(L^\vee)$.

Property $C(L)$

There is a bijection $\mathfrak{L}_L\colon \mathrm{Irr}_{\mathrm{cusp}}(L) \xrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)$.

Property $C^+(L)$

There is a bijection $\mathfrak{L}_L\colon \mathrm{Irr}_{\mathrm{cusp}}(L) \xrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)$ such that

$$\mathfrak{L}_L \circ \mathrm{Ad}(w) = \mathrm{Ad}(w^\vee) \circ \mathfrak{L}_L \quad \text{for any } w \in W_G(L).$$

Remark

Property $C^{+}(L)$ for $\mathfrak{L}_L = \mathrm{LLC}_L$ is equivalent to the Cuspidality Conjecture for L.

Remark

When $G = \mathrm{GL}_n(F)$, Property $C^{+}(L)$ is implied by the condition LLC^{+} defined by Haines, and proved by Henniart. In general, Property $C^{+}(L)$ may be viewed as an "enhancement" of (a special case of) LLC^{+}.

Remark

Property $C^{+}(G)$ coincides with Property $C(G)$, and the cuspidality conjecture says that Property $C(G)$ is satisfied with $\mathfrak{L}_G = \mathrm{LLC}$.

Proposition

If Property $C^+(L)$ is satisfied for L a Levi subgroup of G, then $\mathfrak{L}_L$ induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L) \stackrel{1-1}{\longleftrightarrow} \Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee).$$

Proof.

Thanks to Property $C^+(L)$, the map $w \mapsto w^\vee$ restricts to a bijection

$$W_G(L,\sigma) := \mathrm{Fix}_{W_G(L)}(\sigma) \longrightarrow W_{G^\vee}(L^\vee, \mathfrak{L}_L(\sigma)) := \mathrm{Fix}_{W_{G^\vee}(L^\vee)}(\mathfrak{L}_L(\sigma)),$$

and hence induces a bijection $\tau \mapsto \tau^\vee$ from $\mathrm{Irr}(W_G(L,\sigma))$ to $\mathrm{Irr}(W_{G^\vee}(L^\vee, \mathfrak{L}_L(\sigma)))$. Thus we get a bijection

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L) & \to & \Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee) \\ W_G(L)\cdot(\sigma,\tau) & \mapsto & W_{G^\vee}(L^\vee)\cdot(\mathfrak{L}_L(\sigma),\tau^\vee) \end{array}.$$

□

Notation

Let $\mathrm{Lev}(G)$ be a set of representatives for the conjugacy classes of Levi subgroups of G.

Theorem [Moussaoui (2017)]

If G is split classical p-adic group there exists a canonical, bijective, commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi_{\mathrm{e}}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_{L\in\mathrm{Lev}(G)} \mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L) & \longrightarrow & \bigsqcup_{L\in\mathrm{Lev}(G)} \Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee) \end{array}.$$

Open question

Check that the theta correspondence for a reductive dual pair (G_1, G_2) preserves Bernstein series, then try to describe it in terms of an explicit correspondence between extended quotients for G_1 and extended quotients for G_2.

Theorem [A-Baum-Plymen-Solleveld (2019)]

If G is an inner form of $\mathrm{GL}_n(F)$ there exists a canonical, bijective, commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi_{\mathrm{e}}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_{L \in \mathrm{Lev}(G)} \mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L) & \longrightarrow & \bigsqcup_{L \in \mathrm{Lev}(G)} \Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee) \end{array} .$$

Theorem [A-Xu] (work in progress towards an expicit LLC for G_2)

If G is the exceptional p-adic group of type G_2 there exists a bijective, commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi_{\mathrm{e}}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_{L\in\mathrm{Lev}(G)} \mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W_G(L) & \longrightarrow & \bigsqcup_{L\in\mathrm{Lev}(G)} \Phi_{\mathrm{e,cusp}}(L)/\!/W_{G^\vee}(L^\vee) \end{array} .$$

Remark

Any proper Levi subgroup L of $\mathrm{G}_2(F)$ is isomorphic to either $F^\times \times F^\times$ (torus) or $\mathrm{GL}_2(F)$. So LLC is known for L. Need to check that $C(L)^+$ holds true.

Brief description of the situation:

- All positive-depth supercuspidal representations of $\mathrm{G}_2(F)$, that are attached to twisted Levi sequences of length at least 2, are regular in Kaletha's sense [A-Xu]: so LLC is known for them (Kaletha).
- $\mathrm{G}_2(F)$ has 4 unipotent supercuspidal representations: LLC is known for them (Reeder, Morris, Lusztig).
- Most of the non-unipotent depth-zero supercuspidal representations of $\mathrm{G}_2(F)$ are non-singular in Kaletha's sense [A-Xu]: so LLC is known for them (Kaletha).
- When $q \equiv -1 \mod 3$, there exists a singular non-unipotent depth-zero supercuspidal representation of $\mathrm{G}_2(F)$, it shares its L-packet with an irreducible representation occurring in the parabolically induced representation from a supercuspidal representation of the Levi $\simeq \mathrm{GL}_2$ associated to the long root [A-Xu].

Question: Is it the general picture?

Expected answer:

- Yes, if G is quasi-split (not proved, no known counter-example...),
- If G is not quasi split: Yes, but up to a possible twist!

Theorem [A-Baum-Plymen-Solleveld, 2019]

If G is an inner form of $\mathrm{SL}_n(F)$ there exists a family of 2-cocycles $\natural$ and a (canonical up to permutations within L-packets) bijective commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi_{\mathrm{e}}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_{L\in\mathrm{Lev}(G)} (\mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L))_\natural & \longrightarrow & \bigsqcup_{L\in\mathrm{Lev}(G)} (\Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee))_\natural \end{array}.$$

An interesting example to study:

The inner form $\mathrm{GU}_2(D)$ of GSp_4, it is the similitude group of the unique 2-dimensional Hermitian vector space over the quaternion division F-algebra D. It is isomorphic as an algebraic group to $\mathrm{GSpin}_{4,1}$, the general spin group associated to the (unique up to scaling) non-split quadratic space of dimension 5 over F.

Theorem [A-Moussaoui-Solleveld (2018)]

For any G, there is a bijection:

$$\Phi_{\mathrm{e}}(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathrm{Lev}(G)} (\Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee))_{L\natural}.$$

Theorem [Solleveld, 2020]

For any G, there is a bijection:

$$\mathrm{Irr}(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathrm{Lev}(G)} (\mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L))_{\natural}.$$

Consequence of the two theorems

If $C^+(L)$ is satisfied for any Levi subgroup of G, then we obtained a bijection

$$\mathrm{Irr}(G) \overset{1-1}{\longleftrightarrow} \Phi_{\mathrm{e}}(G).$$

Open problem

Prove that the bijection above “is” the LLC.

Twisted extended quotients [A-Baum-Plymen-Solleveld]

Let Γ be a group acting on a space X and let Γ_x denote the stabilizer in Γ of $x \in X$. Let $\natural$ be a collection of 2-cocycles

$$\natural_x \colon \Gamma_x \times \Gamma_x \to \mathbb{C}^\times,$$

such that $\natural_{\gamma x}$ and $\gamma_* \natural_x$ define the same class in $H^2(\Gamma_{\gamma x}, \mathbb{C}^\times)$, where $\gamma_* \colon \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma\alpha\gamma^{-1}$. Let $\mathbb{C}[\Gamma_x, \natural_x]$ be the group algebra of Γ_x twisted by $\natural_x$.

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} \colon \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

satisfying the conditions

(a) if $\gamma x = x$, then $\phi_{\gamma,x}$ is conjugation by an element of $\mathbb{C}[\Gamma_x, \natural_x]^\times$;

(b) $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma$ and $x \in X$.

We set

$$\widetilde{X}_\natural := \{(x, \tau) \, : \, x \in X, \, \tau \in \operatorname{Irr} \mathbb{C}[\Gamma_x, \natural_x]\} \, .$$

Define a Γ-action on $\widetilde{X}_\natural$ by $\gamma \cdot (x, \tau) := (\gamma x, \tau \circ \phi_{\gamma,x}^{-1})$. The twisted extended quotient of X by Γ with respect to $\natural$ is defined to be

$$(X/\!/\Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

Thank you very much for your attention!