Dirac Operators and Representation Theory
3. Proof of Connes-Kasparov isomorphism

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Connes-Kasparov isomorphism

- Let G be an almost connected Lie group and K a maximal compact subgroup.
- $M = K\backslash G$ is a Riemannian manifold with a G-invariant metric given by the killing form.
- Assume $M = K\backslash G$ has a G-equivariant spin structure.

Theorem (Connes-Kasparov isomorphism)

The Dirac induction map

$$R(K) \to K_n(C_r^*(G)) \qquad [V] \mapsto \mathrm{ind}_G(D_V).$$

is an isomorphism of abelian groups:

$$K_n(C_r^*(G)) \cong \begin{cases} R(K) & n = \dim M \\ 0 & \textit{otherwise} \end{cases}$$

Aim

- Let G be a complex or real semisimple Lie group.
- Denote by $\widehat{G}$ the set of irreducible unitary representations of G.
- Let $\widehat{G}_t \subset \widehat{G}$ be the subset of tempered representations of G, i.e., the representation occurred in the left regular representation of G in $L^2(G)$.

Apply Fourier transform on G to study

$$\mathrm{ind}_G(D_V) \in K_n(C_r^*(G)).$$

Understand better the relationship between

- K-theory of group C^*-algebras;
- Representation of semi-simple Lie groups.

1. Case of $\mathbb{R}^n$

Fourier Transform

Let $G = \mathbb{R}^n$. Then

$$\widehat{G} = \widehat{G}_t = \mathbb{R}^n$$

Let D be the Dirac operator on $\mathbb{R}^n$. Then

$$\mathrm{ind}_{\mathbb{R}^n}(D) \in K_n(C_r^*(\mathbb{R}^n)).$$

For the abelian group G, the Fourier transform

$$C_r^*(G) \to C_0(\widehat{G})$$

$$f \mapsto \widehat{f} \qquad \widehat{f}(\chi) := \int_G f(g)\chi(g)dg$$

gives rise to isomorphisms:

$$C_r^*(\mathbb{R}^n) \cong C_0(\widehat{\mathbb{R}^n}) = C_0(\mathbb{R}^n).$$

Therefore,

$$K_n(C_r^*(\mathbb{R}^n)) \cong K_n(C_0(\mathbb{R}^n)) \cong \mathbb{Z}.$$

Proposition

Let $D : C_c^\infty(\mathbb{R}^n, S) \to C_c^\infty(\mathbb{R}^n, S)$ be the Dirac operator on $\mathbb{R}^n$. The higher index

$$\mathrm{ind}_{\mathbb{R}^n}(D) \in K_n(C_r^*(\mathbb{R}^n))$$

is the Bott generator of $K_n(C_0(\mathbb{R}^n))$ under the isomorphism

$$K_n(C_r^*(\mathbb{R}^n)) \cong K_n(C_0(\mathbb{R}^n)) \cong \mathbb{Z}.$$

Take the KK-cycle

$$(\mathcal{E}_G(\mathbb{R}^n, S), D)$$

representing $\mathrm{ind}_{\mathbb{R}^n}(D) \in K_n(C_r^*(\mathbb{R}^n))$ and find the image under the Fourier transform

$$K_n(C_r^*(\mathbb{R}^n)) \cong K_n(C_0(\widehat{R^n})).$$

Denote by $H_\pi = \mathbb{C}$ the representation space of $\pi \in \widehat{\mathbb{R}^n}$.

Lemma

The Fourier transform of D on $\mathcal{E}(\mathbb{R}^n, S)$ is identified with $\{D_\pi\}_{\pi \in \widehat{\mathbb{R}^n}}$ on $C_0(\widehat{\mathbb{R}^n}, \{H_\pi \otimes S\}_{\pi \in \widehat{\mathbb{R}^n}})$ where

$$D_\pi(\lambda \otimes s) = \lambda \otimes c(\pi)s \qquad \lambda \in H_\pi, s \in S.$$

Proof: Regard $\mathcal{E}_G(\mathbb{R}^n, S)$ as a direct summand of the free module

$$C_r^*(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n, S)$$

(with action of $1 \otimes D$). Under the Fourier transform:

$$f \in C_r^*(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n, S) \cong C_0(\widehat{\mathbb{R}^n}, L^2(\widehat{\mathbb{R}^n}, S)) \ni \widehat{f}$$

where

$$\widehat{f}(\pi) = \int_{\mathbb{R}^n} \pi(g)\widehat{f(g)}dg \in C_0(\widehat{\mathbb{R}^n}, L^2(\widehat{\mathbb{R}^n}, S)).$$

The property of Fourier transform implies that

$$\widehat{Df}(\pi) = D_\pi \widehat{f}(\pi) \qquad \pi \in \widehat{\mathbb{R}^n}.$$

Here D_π for $\pi \in \widehat{\mathbb{R}^n}$ is the scalar multiplication by $\sigma_D = ic(\xi)$ on $L^2(\widehat{\mathbb{R}^n}, S)$ over the fiber $\pi \in \widehat{\mathbb{R}^n}$:

$$[D_\pi f](\xi) = ic(\xi)f(\xi) \qquad f \in L^2(\widehat{\mathbb{R}^n}, S).$$

The Fourier transform of $\mathcal{E}_G(\mathbb{R}^n, S)$ has the form

$$C_0(\widehat{\mathbb{R}^n}, \{H_\pi \otimes S\}_{\pi \in \widehat{\mathbb{R}^n}})$$

and the restriction of the family of operators $\{D_\pi\}_{\pi \in \widehat{\mathbb{R}^n}}$ to the above space is

$$(C_0(\widehat{\mathbb{R}^n}, S), \{ic(\pi)\}_{\pi \in \widehat{R^n}})$$

The lemma is then proved.
Observe that this is exactly the Bott element of $KK_n(\mathbb{C}, C_0(\mathbb{R}^n))$.

Alternative Interpretation

One can work directly on the Dirac operator D on

$$L^2(\mathbb{R}^n, S) = L^2(\mathbb{R}^n) \otimes \Delta,$$

which represents a cycle in $K_n(\mathbb{R}^n)$ or $K_n(C_0(\mathbb{R}^n))$, and find the Fourier transform under the Plancherel decomposition:

$$L^2(\mathbb{R}^n) = \int_{\widehat{\mathbb{R}^n}} H_\pi \otimes H_\pi^* d\mu(\pi)$$

and D is decomposed into

$$D_\pi : H_\pi \otimes (H_\pi^* \otimes \Delta) \to H_\pi \otimes (H_\pi^* \otimes \Delta)$$

given by the Clifford multiplication by $c(\pi)$ on Δ for $\pi \in \widehat{\mathbb{R}^n}$. This is the same family given by the Bott element

$$(C_0(\widehat{\mathbb{R}^n}, S), \{ic(\pi)\}_{\pi \in \widehat{R^n}}) \in KK_n(\mathbb{C}, C_0(\widehat{\mathbb{R}^n})).$$

2. Case of complex semisimple Lie groups

Set up

- Let G be a complex connected semisimple Lie group (for example, $SL(2,\mathbb{C})$) and K a maximal compact subgroup.
- Let $V = V_\tau \in \widehat{K}$ where $\tau \in i\mathfrak{t}$ is the highest weight of V in the positive Weyl chamber of K. So $[V] \in R(K)$.

Compute

$$\mathrm{ind}_G(D_{V_\tau}) = [((L^2(G) \otimes \Delta \otimes V_\tau)^K, D_{V_\tau})] \in KK_n(\mathbb{C}, C_r^*(G))$$

where $n = \dim(G/K) = \dim A$.

Theorem (Penington-Plymen)

The Dirac induction is an isomorphism

$$R(K) \to K_n(C_r^*(G)) \qquad n = \dim(G/K)$$

sending $E = E_\tau \in \widehat{K}$ with highest weight τ to the component for $\sigma \in \widehat{M}$ with highest weight $\tau + \rho_K$, where

- *ρ_K is the half sum of compact positive roots.*

To treat a component of a family of irreducible principal series, need to:

- Decompose $C_r^*(G)$ using Fourier transformation;
- Decompose $(L^2(G) \otimes \Delta \otimes V_\tau)^K$ using Plancherel formula for $L^2(G)$;
- Find the Fourier transform of D at each $(\pi, H_\pi) \in \widehat{G_t}$;
- Study properties of D_π and $(H_\pi \otimes \Delta \otimes V_\tau)^K$. In fact
  - D_π^2 is a scalar multiplication;
  - D_π does not contribute to the family index if D_π^2 is positive;
  - $(H_\pi \otimes \Delta \otimes V_\tau)^K$ has finite dimension and vanishes in many occasions.
- Conclude that $\{D_{\sigma,\lambda}\}_{\lambda \in \widehat{A}}$ is a Bott generator on the $\widehat{A}$ component over some $\sigma \in \widehat{M}$.

Decompose $C_r^*(G)$

Under the Fourier transform, $C_r^*(G) \cong C_0(\widehat{G_t}, \mathcal{K})$ for a complex semisimple Lie group.
Denote by the connected component of $\widehat{G_t} = (\widehat{M} \times \widehat{A})/\mathbb{Z}_2$ by

$$C_\sigma = \begin{cases} \{\sigma\} \times \widehat{A} & W_\sigma = 1 \\ \{\sigma\} \times (\widehat{A}/W_\sigma) & W_\sigma \neq 1 \end{cases}$$

Then

$$C_r^*(G) \cong \bigoplus_{\sigma \in \widehat{M}/W} C_0(C_\sigma, \mathcal{K}(H_\sigma)), \qquad H_\sigma = \mathrm{Ind}_P^G C_0(\widehat{A}, V_\sigma).$$

- V_σ is the representation space of $\sigma \in \widehat{M}$;
- $C_0(\widehat{A}, V_\sigma)$ is a representation space for $\widehat{A} \times \{\sigma\}$;
- Every tempered irreducible representation of G has the form $(\pi_{\sigma,\lambda}, H_{\sigma,\lambda}) = \mathrm{Ind}_P^G \sigma \otimes \lambda \otimes 1$ for $\sigma \in \widehat{M}, \lambda \in \widehat{A}$ and

$$H_\sigma = \int_{C_\sigma}^{\oplus} H_{\sigma,\lambda} d\mu(\lambda).$$

Decompose $L^2(G)$

Apply Plancherel decomposition of $L^2(G)$, one has

$$\begin{aligned}(L^2(G)\otimes\Delta\otimes V_\tau)^K &= \bigoplus_{\sigma\in\widehat{M}/W}\int_{C_\sigma}(H_{\sigma,\lambda}\otimes H^*_{\sigma,\lambda}\otimes\Delta\otimes V_\tau)^K\\ &= \bigoplus_{\sigma\in\widehat{M}/W}\int_{C_\sigma}H_{\sigma,\lambda}\otimes(H^*_{\sigma,\lambda}\otimes\Delta\otimes V_\tau)^K.\end{aligned}$$

Lemma

$$\dim(H^*_{\sigma,\lambda}\otimes\Delta\otimes V_\tau)^K=\begin{cases}2^{\lfloor\frac{1}{2}\dim A\rfloor} & \sigma=\tau+\rho_K\\ 0 & \tau+\rho_K-\sigma\notin \textit{ pos. Weyl cham.}\end{cases}$$

Idea of proof: Let $W_a\in\widehat{K}$ be with highest weight $a\in i\mathfrak{t}^*$. Note that $H_{\sigma,\lambda}|_K=(\mathrm{Ind}_P^G\sigma\otimes\lambda\otimes 1)|_K=\mathrm{Ind}_M^K\sigma$. Then

$$H_{\sigma,\lambda}|_K=W_\sigma\oplus W_{<\sigma}\qquad \Delta=2^{\lfloor n/2\rfloor}W_{\rho_K}\oplus W_{<\rho_K}\qquad V_\tau=W_\tau.$$

The Lemma follows as a result of Schur's lemma.

Fourier transform D_{V_τ}

For a Dirac operator D on $(L^2(G) \otimes \Delta \otimes V_\tau)^K$, denote by $D_{\sigma,\lambda}$ its restriction to the component $H_{\sigma,\lambda} \otimes (H^*_{\sigma,\lambda} \otimes \Delta \otimes V_\tau)^K$:

$$\widehat{Df}(\pi_{\sigma,\lambda}) = D_{\sigma,\lambda}\widehat{f}(\pi_{\sigma,\lambda})$$

for $f \in (L^2(G) \otimes \Delta \otimes V_\tau)^K$ and $\pi_{\sigma,\lambda} \in \widehat{G_t}$.

Lemma

For $D = D_{V_\tau}$, the square of $(D_{V_\tau})_{\sigma,\lambda}$ is a scalar given by

$$(D_{V_\tau})^2_{\sigma,\lambda} = -\|\sigma\|^2 + \|\lambda\|^2 + \|\tau + \rho_K\|^2.$$

Used the Parthatherathy's formula for $D^2_{V_\tau}$.

Corollary

If $\tau + \rho_K - \sigma$ is in the positive Weyl Chamber, then $(D_{V_\tau})^2_{\sigma,\lambda} > 0$ and hence $(D_{V_\tau})_{\sigma,\lambda}$ is invertible. ($\|\cdot\|$ is defined by the inner product on $i\mathfrak{t}^*$.)

Simplification

Under Fourier transform:

$$\mathrm{ind}_G(D_{V_\tau}) = [((L^2(G)\otimes\Delta\otimes V_\tau)^K, D_{V_\tau})] \in KK_n(\mathbb{C}, C_r^*(G))$$

is decomposed into a direct sum over $\widehat{M}/W$:

$$\left[\left(\int_{C_\sigma} H_{\sigma,\lambda}\otimes(H_{\sigma,\lambda}^*\otimes\Delta\otimes V_\tau)^K d\mu(\lambda), \{(D_{V_\tau})_{\sigma,\lambda}\}_{\lambda\in\widehat{A}}\right)\right]$$

in $KK_n(\mathbb{C}, C_0(\widehat{A}, \mathcal{K}(H_\sigma)))$.
Using the two key lemmas, only the component for $\sigma = \tau + \rho_K$ will survive. And from this we also know that $C_\sigma = \widehat{A}$:

Lemma

The component for $\sigma \in \widehat{M}$ is nonzero in K-theory if and only if there exists $w \in W$ such that the highest weight of $w\sigma$ has the form $\tau + \rho_K$ for some τ in the positive Weyl chamber of for K.

Therefore, $[((L^2(G)\otimes\Delta\otimes V_\tau)^K, D_{V_\tau})]$ is simplified to

$$\left[\left(\int_{\widehat{A}} H_{\tau+\rho_K,\lambda}\otimes(H^*_{\tau+\rho_K,\lambda}\otimes\Delta\otimes V_\tau)^K d\mu(\lambda), \{(D_{V_\tau})_{\tau+\rho_K,\lambda}\}_{\lambda\in\widehat{A}}\right)\right].$$

Denote $\mathbb{S}=(H^*_{\tau+\rho_K,\lambda}\otimes\Delta\otimes V_\tau)^K$, having dim $2^{\lfloor \dim A/2\rfloor}$, and

$$(D_{V_\tau})_{\tau+\rho_K,\lambda}\in\mathrm{End}(H_{\tau+\rho_K,\lambda}\otimes\mathbb{S})$$

as $C_0(\widehat{A},\mathcal{K}(H_{\tau+\rho_K}))$-modules, that is, $(D_{V_\tau})_{\tau+\rho_K,\lambda}$ commutes with $\mathcal{K}(H_{\tau+\rho_K,\lambda})$. Thus, $(D_{V_\tau})_{\tau+\rho_K,\lambda}$ has the form

$$1\otimes M_{\tau,\lambda} \qquad M_{\tau,\lambda}\in\mathrm{End}(\mathbb{S}).$$

Because $(D_{V_\tau})^2_{\tau+\rho_K,\lambda}=\|\lambda\|^2$,

$$(D_{V_\tau})_{\tau+\rho_K,\lambda}=1\otimes ic(\lambda).$$

Then under Morita equivelence $C_0(\widehat{A},\mathcal{K}(H_{\tau+\rho_K}))\sim C_0(\widehat{A})$, we have the Bott generator:

$$\left[\left(\int_{\widehat{A}} H_{\tau+\rho_K,\lambda}\otimes\mathbb{S}, 1\otimes ic(\lambda)\right)\right]=\left[\left(\int_{\widehat{A}}\mathbb{S}d\mu(\lambda), ic(\lambda)\right)\right]$$

3. Discrete series and the examples of $SL(2, \mathbb{R})$

Discrete Series

Theorem (Wassermann, Clare-Crisp-Higson)

Let G be a connected real semisimple Lie group. The Fourier transform gives rise to an isomorphism

$$C_r^*(G) \cong \bigoplus_{[(P,\sigma)]} C_0(\mathfrak{a}_P^*, \mathcal{K}(\mathrm{Ind}_P^G(H_\sigma)))^{W_\sigma}.$$

For a discreet series component, we have $P = G$, $\sigma \in \widehat{G_d}$, the component $C_0(\mathfrak{a}_P^*, \mathcal{K}(\mathrm{Ind}_P^G(H_\sigma)))^{W_\sigma}$ reduces to $\mathcal{K}(H_\sigma)$ and

$$\mathbb{Z} \cong K_0(\mathcal{K}(H_\sigma)) \leq K_0(C_r^*(G)).$$

Discrete series of $SL(2,\mathbb{R})$

- Let $G = SL(2,\mathbb{R})$ and $K = SO(2)$.
- Let V_k be irreducible representation of K with weight $k \in \mathbb{Z}$.

Consider the Dirac operator

$$D_{V_k} : (L^2(G) \otimes \Delta \otimes V_k)^K \to (L^2(G) \otimes \Delta \otimes V_k)^K.$$

When $k \neq 0$, as in the complex semisimple case, the principle series component of the image of the Dirac induction vanishes

$$\mathrm{ind}_G(D_{V_k}) = [((L^2(G) \otimes \Delta \otimes V_k)^K, D_{V_k})] \in K_0(C_r^*(G)).$$

$$K_0(C_r^*(G)) \cong K_0(C_0(\mathbb{R}) \rtimes \mathbb{Z}_2) \bigoplus (\oplus_{n\neq 0} K_0(\mathcal{K}(H_n))).$$

Here $H_n := D_n^+$ if $n > 0$ and $H_n := D_n^-$ if $n < 0$.
Under the Fourier transform, $\mathrm{ind}_G(D_{V_k}) \in K_0(C_r^*(G))$ reduces to

$$\oplus_{n\neq 0}(H_n \otimes (H_n^* \otimes \Delta \otimes V_k)^K, (D_{V_k})_n) \in \oplus_{n\neq 0} K_0(\mathcal{K}(H_n)).$$

Discrete series of $SL(2,\mathbb{R})$

Under Morita equivalence

$$\oplus_{n\neq 0}((H_n^*\otimes\Delta\otimes V_k)^K, M_{k,n})\in\oplus_{n\neq 0}K_0(\mathbb{C}).$$

Because $(H_n^*\otimes\Delta\otimes V_k)^K$ has finite dimension and $\mathbb{Z}_2$-graded, the image in $K_0(\mathbb{C})\cong\mathbb{Z}$ can be calculated by the index

$$\oplus_{n\neq 0}[\dim(H_n^*\otimes\Delta^+\otimes V_k)^K-\dim(H_n^*\otimes\Delta^-\otimes V_k)^K]\in\oplus_{n\neq 0}\mathbb{Z}$$

The number

$$\dim(H_n^*\otimes\Delta\otimes V_k)^K-\dim(H_n^*\otimes V^-\otimes V_k)^K=\begin{cases}1 & n=k\\ 0 & n\neq k\end{cases}$$

So

$$\mathrm{ind}_G(D_{V_k})=[H_k]\in K_0(C_r^*(G))\qquad k\neq 0.$$

- How to identify $[V_0]$ with $K_0(C_0(\mathbb{R})\rtimes\mathbb{Z}_2)$ by FT? (Brodzki-Niblo-Plymen-Wright, Clare-Higson-Song-Vogan)

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