Dirac Operators and Representation Theory
2. Connes-Kasparov isomorphism

Hang Wang

East China Normal University

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1. Index in $K_n(C_r^*(G)$

K-theoretic Index

Let M be a complete Riemannian manifold acted properly, isometrically and cocompactly by a locally compact group. Let D be a G-invariant Dirac type operator on $C_c^\infty(M,S)$. Then

- D is a regular ess. s.a. operator on some $C_r^*(G)$-module $\mathcal{E}_G$. So
$$C_0(\mathbb{R}) \to \mathcal{K}(\mathcal{E}_G) \qquad f \mapsto f(D).$$
- Let $\chi : \mathbb{R} \to (-1,1)$ be an odd increasing function such that $\lim_{x\to\infty} \chi(x) = 1$. Then $\chi(D)^2 - I \in \mathcal{K}(\mathcal{E}_G)$.

Definition

The higher index or the equivariant index of D (odd, $\mathbb{Z}_2$-graded) is

$$\mathrm{ind}_G(D) := \partial[\chi(D)] \in K_0(\mathcal{K}(\mathcal{E}_G)) \cong K_0(C_r^*(G)).$$

Here $\partial : K_1(\mathcal{B}(\mathcal{E}_G)/\mathcal{K}(\mathcal{E}_G)) \to K_0(\mathcal{K}(\mathcal{E}_G))$ is the boundary map of the six-term exact sequence associated to

$$0 \to \mathcal{K}(\mathcal{E}_G) \to \mathcal{B}(\mathcal{E}_G) \to \mathcal{B}(\mathcal{E}_G)/\mathcal{K}(\mathcal{E}_G) \to 0.$$

Remark

- $\mathcal{K}(\mathcal{E}_G)$ is Morita equivalent to $C_r^*(G)$.
- In the ungraded case or in the case of a canonical Dirac operator when M has odd dimension, one can similarly define a higher index

$$\mathrm{ind}_G(D) \in K_1(\mathcal{E}_G) \cong K_1(C_r^*(G)).$$

Example

When $M = K \backslash G$, $S = G \times_K \Delta$, and if G is abelian:

$$\mathcal{K}(\mathcal{E}_G) \subset C_r^*(G) \otimes \mathrm{End}(\Delta).$$

We shall see that

$$\mathrm{ind}_G(D) \in K_n(C_r^*(G)) \cong K_n(C_0(\widehat{G}))$$

is the index of a family of Fredholm operators $\{D_\pi\}_{\pi \in \widehat{G}}$.

Higher Index as a KK-cycle

For the purpose of a proof, it would be more convenient to view $\mathrm{ind}_G(D) \in KK(\mathbb{C}, C_r^*(G))$, represented by the unbounded Kasparov cycle

$$[(\mathcal{E}_G, D)] \in KK_i(\mathbb{C}, C_r^*(G)).$$

- When S has a $\mathbb{Z}_2$-grading and D is odd, $i = 0$;
- When S has no grading, $i = 1$.

Remark

- The isomorphism $KK_i(\mathbb{C}, C_r^*(G)) \cong K_i(C_r^*(G))$ is given by the "index map" $[(\mathcal{E}_G, D)] \to \mathrm{ind}_G(D)$.
- In the even (graded) case, $[(\mathcal{E}_G(M,S), D)]$ can be written as

$$[(\mathcal{E}_G(M,S^+), \mathcal{E}_G(M,S^-), (1 + D^+D^-)^{-\frac{1}{2}}D^+)] \in K_0(C_r^*(G)),$$

the alternative picture of $K_0(A)$ by operators on A-modules.

Example

Bott element $\beta = \frac{1}{1+|z|^2}\begin{bmatrix} 1 & \bar{z} \\ z & |z|^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in K_0(C_0(\mathbb{R}^2))$.

Corresponding element in $KK(\mathbb{C}, \mathbb{C}_0(\mathbb{R}^2))$ is represented by

$$\left(C_0(\mathbb{R}^2, \underline{\mathbb{C}^2}), \begin{bmatrix} 0 & \bar{z} \\ z & 0 \end{bmatrix}\right) \quad \text{or} \quad \left(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2), \frac{z}{1+|z|^2}\right).$$

Proof.

Let $\chi(x) = \frac{x}{\sqrt{1+|x|^2}}$, and $D = \begin{bmatrix} 0 & \bar{z} \\ z & 0 \end{bmatrix}$. Then

$\chi(D) = \begin{bmatrix} 0 & \frac{\bar{z}}{1+|z|^2} \\ \frac{z}{1+|z|^2} & 0 \end{bmatrix}$ Lift this to an invertible element in

$M_2(C_b(\mathbb{R}^2))$: $L = \begin{bmatrix} \frac{1}{|1+|z|^2|} & \frac{\bar{z}}{1+|z|^2} \\ \frac{z}{1+|z|^2} & \frac{1}{1+|z|^2} \end{bmatrix}$ and then

$$\partial[\chi(D)] = L\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}L^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \beta$$

Bott generator

Proposition

The generator for $K_n(C_0(\mathbb{R}^n))$ is given by

$$\left(C_0(\mathbb{R}^n, \underline{\mathbb{C}^{2^{\lfloor n/2 \rfloor}}}), ic(x)\right) \in KK_n(\mathbb{C}, C_0(\mathbb{R}^n)).$$

Here, $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \mathbb{R}^n$.

Example

When $n = 2$:

$$\begin{bmatrix} 0 & \bar{z} \\ z & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{bmatrix} = ix_1 \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + ix_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= ix_1 c(e_1) + ix_2 c(e_2) = ic(x).$$

Remark

This presentation of the Bott element is often used in E-theory.

2. Connes-Kasparov isomorphism, examples

Connes-Kasparov isomorphism

- G: almost connected Lie group, K: maximal compact
- $M = K\backslash G$: Riemannian manifold with a G-invariant metric.

Theorem (Connes-Kasparov isomorphism)

Assume $M = K\backslash G$ has a G-equiv spin structure. The Dirac induction map

$$R(K) \to K_n(C_r^*(G)) \qquad [V] \mapsto \mathrm{ind}_G(D_V).$$

is an isomorphism of abelian groups:

$$K_n(C_r^*(G)) \cong \begin{cases} R(K) & n = \dim M \\ 0 & \textit{otherwise} \end{cases}$$

- If $K\backslash G$ has no G-equivariant spin structure, replace $R(K)$ by

$$R^1(K) = \{\pi \in R(\tilde{K}) \text{ that factors through } K\}.$$

$\tilde{K}$ is the double cover of K.

Let G be a complex semisimple Lie group. Then

$$\widehat{G_t} = (\hat{M} \times \hat{A})/W$$

is Hausdorff (A, M are in the Borel subgroup $P = MAN$). Then

$$\begin{aligned} K_i(C_c^*(G)) =& K^i(G_t) = K^i((\hat{M} \times \hat{A})/W) \\ =& \bigoplus_{W_\sigma = 1} K^i(\{\sigma\} \times \hat{A}) \qquad i = \dim A \end{aligned}$$

Theorem (Penington-Plymen)

The Dirac induction is an isomorphism

$$R(K) \to K_i(C_r^*(G))$$

sending $E = E_\tau \in \widehat{K}$ with highest weight τ to the component for $\sigma \in \hat{M}$ with highest weight $\tau + \rho_K$, where

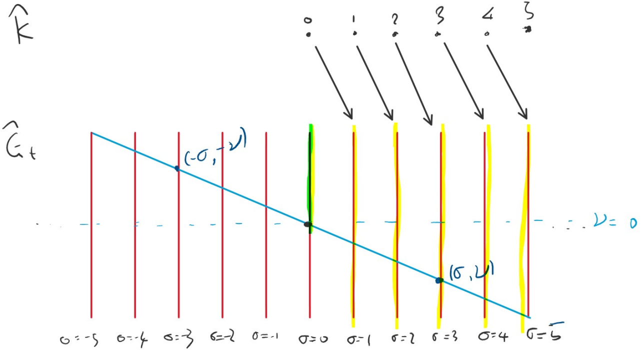
- *ρ_K is the half sum of compact positive roots.*

Example

Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$. $\dim M = 3$.

$$R(SU(2)) = \bigoplus_{n \geq 0} \mathbb{Z};$$

$$\begin{aligned} K_*(C_r^*(SL(2,\mathbb{C}))) &\cong K^*((\hat{M} \times \hat{A})/W) \\ &\cong K^*((\mathbb{Z} \times \mathbb{R})/\mathbb{Z}_2) = \begin{cases} \bigoplus_{n>0} \mathbb{Z} & * = 1 \\ 0 & * = 0 \end{cases} \end{aligned}$$



Semisimple Lie with one conj. class of Cartan subgroups

Let G be a semisimple Lie group having one conjugacy class of Cartan subgroups. This include the complex semisimple case, but also the example of $SL(3, \mathbb{R})$. In this setting $\widehat{G_t}$ remains Hausdorff and

$$\widehat{G_t} \cong (\hat{M} \times \hat{A})/W$$

Theorem (Valette)

The Dirac induction is an isomorphism

$$R(K) \to K_i(C_r^*(G))$$

sending $E = E_\tau \in \widehat{K}$ with highest weight τ to the component for $\sigma \in \hat{M}$ with highest weight $\tau + \rho_K - \rho_M$, where

- *ρ_K is the half sum of compact positive roots;*
- *ρ_M is the half sum of positive root of M.*

Semisimple case having discrete series

Let G be a connected semisimple Lie group with discrete series. Let π_λ be the discrete series with Harich-Chandra parameter $\lambda \in i\mathfrak{t}^*$, i.e.,

- $(\lambda, \alpha) \neq 0$ for all $\alpha \in R(\mathfrak{t}, \mathfrak{g})$;
- $\lambda + \rho^\lambda$ analytic integral.

Theorem (Atiyah-Schmid)

The Dirac induction

$$R(K) \to K_0(C_r^*(G))$$

sends $E = E_\tau \in \widehat{K}$ with highest weight τ to the discrete series generator with Harish-Chandra parameter $\tau + \rho_K$ where

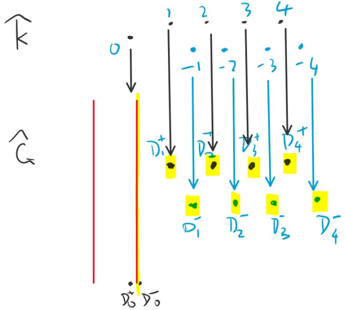
- *ρ_K is the half sum of compact positive roots.*

Example

Let $G = SL(2,\mathbb{R})$ and $K = SO(2)$. $\dim M = 2$.

$$R(SO(2)) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z};$$

$$K_0(C_r^*(SL(2,\mathbb{R}))) \cong \oplus_{n\neq 0}\mathbb{Z} \bigoplus K_0(C_0(\mathbb{R}) \rtimes \mathbb{Z}_2) \oplus K_0(C_0(\mathbb{R}/\mathbb{Z}_2))$$



- $K^0_G(M)$ the Grothendieck group generated by the G-equivariant vector bundles over M.
- Homogeneous vector bundles has the form $\mathcal{V} = G \times_K V$.
- The induction from a K-representation V to a $\mathcal{V}$ gives rise to an isomorphism

$$R(K) \to K^0_G(M) \qquad [V] \mapsto [\mathcal{V}] := [K \backslash V \times G].$$

When M admits a G-equivariant spin structure, one has the twisted Dirac operator $D_{\mathcal{V}}$ and by taking the higher index:

$$\nu : K^0_G(M) \to K_n(C^*_r(G)) \qquad [\mathcal{V}] \mapsto [\mathrm{ind}_G(D_{\mathcal{V}})].$$

- Connes-Kasparov isomorphism is equivalent to μ being an isomorphism;
- ν is a special case of the Baum-Connes assembly map.

2. Relation to the Baum-Connes conjecture

- G: locally compact group; M: spin Riemannian manifold;
- G acts on M properly, cocompactly, isometrically.

The higher index map gives rise to a morphism

$$K^0_G(M) \to K_m(C^*_r(G)) \qquad [E] \to \mathrm{ind}_G(D_E).$$

Assume

- M has G-equivariant spin structure;
- G is a discrete or a Lie group

one has the Poincaré duality

$$K^0_G(M) \cong K^G_n(M) \quad [E] \mapsto [D_E].$$

Composing with ν one obtains a morphism

$$\mu : K^G_n(M) \to K_n(C^*_r(G)) \qquad [F] \mapsto \mathrm{ind}_G(F).$$

Remark

μ can be defined without assuming M being spin.

Universal Space for Proper Actions

Assume in addition that proper G-space M is universal: For every proper G-space X there is a unique (up to homotopy) G-equivariant map

$$X \to M.$$

Locally, a proper G space is covered by G-slices of the form $G \times_H U$ where $U \cong \mathbb{R}^k$ is a H-space ($H \subset G$: compact subgroup).

Example

When G is an almost connected Lie group, $K \backslash G$ is universal. The G-equivariant map locally has the form:

$$G \times_H U \to K \backslash G.$$

Proposition

If M is universal, then M is contractible.

Because M is unique up to homotopy, denote M by $\underline{E}G$, called the universal space of proper actions.

Baum-Connes isomorphism

In general, $\underline{E}G$ is CW complex rather than a manifold, and is not always G-cocompact.
Denote by

$$K_n^G(\underline{E}G) = lim_{X \to \underline{E}G} K_n^G(X)$$

where the limit is taken for all proper cocompact manifold X and $f : X \to \underline{E}G$ is a G-equivariant map.

Definition

The Baum-Connes assembly map

$$\mu : K_n^G(\underline{E}G) \to K_n(C_r^*(G))$$

is given by $f_*[F] \to \mathrm{ind}_G(F)$ where $[F] \in K_n^G(X)$ is represented by a G-invariant abstract elliptic operator F.

Conjecture (Baum-Connes)

The Baum-Connes assembly map μ is an isomorphism.

Corollary

The Baum-Connes conjecture is proved to be true for an almost connect Lie group and hence implies the Connes-Kasparov isomorphism.

Theorem (Connes-Kasparov isomorphism)

When G is an almost connected Lie group, the Baum-Connes assembly map

$$\mu : K_n^G(K\backslash G) \to K_n(C_r^*(G))$$

is an isomorphism. This gives rise to the isomorphism of the Dirac induction map

$$R^1(K) \to K_n(C_r^*(G)) \qquad [V] \to [\mathrm{ind}_G(D_V)].$$

The name Connes-Kasparov isomorphism appears first in an early work by Plymen. Logically, it is a combination of

- Connes' Thom isomorphism for crossed product by $\mathbb{R}^n$;
- Kasparov's Dirac-dual Dirac method in showing the Novikov conjecture for almost connected Lie groups.

4. From Connes' Thom isomorphism to Mackey machine

Connes' Thom isomorphism

Theorem (Connes)

Let A be a C^*-algebra carrying an action by $\mathbb{R}^n$. Then

$$K_i(A \rtimes \mathbb{R}^n) \cong K_{i+n}(A).$$

In particular, when the action is trivial, we have

$$A \rtimes \mathbb{R}^n \cong A \otimes C^*(\mathbb{R}^n) \cong A \otimes C_0(\widehat{\mathbb{R}^n}) = A \otimes C_0(\mathbb{R}^n)$$

Corollary (Bott periodicity)

$$K_i(A \otimes C_0(R^{2n})) \cong K_i(A).$$

The Bott map $K_0(A) \to K_0(A \otimes C_0(\mathbb{R}^2))$ is defined by tensoring the Bott element on $\mathbb{R}^2$:

$$\beta := \frac{1}{|z|^2+1}\begin{bmatrix} |z|^2 & \bar{z} \\ z & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Connected semisimple Lie group

- Let G be a connected Lie group with a maximal compact subgroup K;
- Observe that $K\backslash G$ is diffeomorphic to Euclidean space $\mathbb{R}^n$ ($n = \dim K\backslash G$).

Question: Is there an isomorphism

$$K_i(A \rtimes G) \cong K_{i+n}(A \rtimes K)?$$

If so, this reduces to the Connes-Kasparov isomorphism when $A = \mathbb{C}$.

Remark

The problem is difficult for the case when A is an arbitrary C^*-algebra. If $A = C_0(G/\Gamma)$ for a discrete subgroup of G, then this will imply the Baum-Connes conjecture for Γ.

Equivariant Bott periodicity

- Let x_0 be class of $1 \in G$ in $K\backslash G$. Then $K = G_{x_0}$.

Reformulate the conjecture by introducing the Euclidean space

$$V = T_{x_0}(G/K) = \mathfrak{g}/\mathfrak{k}$$

equipped with the action by K and conjecture

$$K_i(A \rtimes_r G) \cong K_i(A \otimes C_0(V) \rtimes K).$$

In particular, when $A = \mathbb{C}$, this reduces to the Connes-Kasparov isomorphism:

$$K_i(C_r^*(G)) \cong K_i(C_0(V) \rtimes K) \cong K_i^K(C_0(V)) \cong \begin{cases} R^1(K) & i-n \text{ even} \\ 0 & i-n \text{ odd} \end{cases}.$$

Under Fourier transform

$$C_0(V) \rtimes K \cong C^*(V) \rtimes K = C^*(V \rtimes K),$$

the Connes-Kasparov isomorphism is equivalent to

$$K_i(C_r^*(G)) \cong K_i(C^*(V \rtimes K)).$$

Let $G_0 = V \rtimes K$, this is known as the motion group.
The Mackey machine predicts a 1-1 correspondence:

$$\widehat{G} \leftrightarrow \widehat{G_0}.$$

K-theoretic Mackey correspondence

It turns out that the Connes-Kasparov isomorphism is a K-theoretic version of the Mackey correspondence.

Theorem (Higson)

Let G be a complex semisimple Lie group. The Mackey machine induces an isomorphism on K-theory

$$K_i(C_r^*(G)) \cong K_i(C_r^*(G_0)).$$

- General cases by Afgoustidis, Higson-Román, Subag, Tan-Yao-Yu, etc.

This K-theoretic isomorphism is closely related to

- the description of the representation of a semisimple Lie group via $(\mathfrak{g}, K)$-modules
- Connes tangent groupoid approach to index theorems.

5. Dirac-dual Dirac method

Dirac induction from Dirac element

Kasparov used KK-theory to formulate the Dirac induction by introducing a Dirac element

$$[D] \in KK^n_G(C_0(K\backslash G), \mathbb{C})$$

constructed from a Dirac type operator D on some homogeneous vector bundle $G \otimes_K S$. The descend homomorphism:

$$j^G : KK^n_G(C_0(K\backslash G), \mathbb{C}) \to KK(C_0(K\backslash G) \rtimes G, C^*_r(G)).$$

Because $C^*(K\backslash G) \rtimes G$ and $C^*(K)$ are Morita equivalent,

$$j^G[D] \in KK(C^*_r(K), C^*_r(G)).$$

Proposition

The Dirac induction map is obtained from KK-product with the the Dirac element $j^G[D]$:

$$R(K) \cong K_0(C^*_r(K)) \to K_n(C^*_r(G)) \qquad [E] \mapsto [E] \otimes_{C^*(K)} j^G[D].$$

Dirac-dual Dirac method

The idea of Dirac-dual Dirac method for showing the Connes-Kasparov isomorphism is to produce a KK-element $\beta \in KK^n_G(\mathbb{C}, C_0(K\backslash G))$ inverse to the Dirac element in the sense of KK product

$$[D] \otimes \beta = 1 \qquad \beta \otimes [D] = 1.$$

Thus, tensoring $j^G(\beta) \in KK^n(C^*_r(G), C^*_r(K))$ gives rise to the inverse to the Dirac induction:

$$\otimes j^G(\beta) : K_n(C^*_r(G)) \to K_0(C^*_r(K))$$

Remark

- Dirac-dual Dirac method is one way of showing the Baum-Connes conjecture.
- Usually, the complication is to construct the Bott element β, so that
$$[D] \otimes \beta = 1.$$
- This implies that
$$\gamma = \beta \otimes [D] \in KK_G(\mathbb{C}, \mathbb{C})$$
is an idempotent with respect to ring structure given by Kasparov product.
- The construction of β or existence of the γ element implies the injectivity of the assembly map, known as the Novikov conjecture.
- Showing $\gamma = 1$ in $KK_G(\mathbb{C}, \mathbb{C})$ will imply that the Baum-Connes assembly map is an isomorphism.

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