Dirac Operators and Representation Theory
1. Dirac Operators

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Outline

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1. What is a Dirac Operator?

Dirac type operators

- Let M be a Riemannian manifold.
- Let $S \to M$ be a Hermitian vector bundle.
- $C_c^\infty(M, S)$=\{compactly supported smooth sections of S\}.

Definition

A Dirac type operator

$$D : C_c^\infty(M, S) \to C_c^\infty(M, S)$$

is a symmetric first order elliptic differential operator satisfying

$$[D, f]^2 = -\|df\|^2 I \qquad f \in C_c^\infty(M).$$

The replacement of df by $[D, f]$ is used to develop

- the notion of distance in the context of a spectral triple;
- the quantized calculus in nonommutative geometry.

Examples

- Let $M = \mathbb{R}^3$. Set
$$D = \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x_i} : C_c^\infty(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{C}^2) \to C_c^\infty(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{C}^2)$$
where
$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$
Check: for $s \in C_c(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{C}^2)$ and $f \in C_c^\infty(\mathbb{R}^2)$
$$[D, f]s = \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x_i}(fs) - \sum_{i=1}^{3} f \sigma_i \frac{\partial}{\partial x_i} s = \sum_{i=1}^{3} \sigma_i \left(\frac{\partial}{\partial x_i} f \right) s.$$
Then using $\sigma_i \sigma_j + \sigma_j \sigma_i = -2\delta_{ij}$, one has
$$[D, f]^2 = \sum_{i,j=1}^{3} \sigma_i \sigma_j f_{x_i} f_{x_j} = -\|df\|^2.$$

Examples

- Let M be an oriented Riemannian manifold and $S = \Lambda^* M$ be the exterior algebra bundle over M. Let $D = d + d^*$ be the de Rham operator on M. With respect to the $\mathbb{Z}_2$-grading

$$S^+ = \Lambda^{even} M \qquad S^- = \Lambda^{odd} M,$$

  D is an odd operator

$$D = \begin{bmatrix} 0 & d + d^*|_{\Lambda^{odd}} \\ d + d^*|_{\Lambda^{even}} & 0 \end{bmatrix} : S^+ \oplus S^- \to S^+ \oplus S^-.$$

- Signature operator
- Dolbeault operator $\bar{\partial} + \bar{\partial}^* : \Lambda^{0,*} M \to \Lambda^{0,*} M$ on a complex manifold M.

Examples

- Given a Dirac type operator $D : C_c^\infty(M,S) \to C_c^\infty(M,S)$ and a vector bundle $V \to M$, there exists a vector bundle $W \to M$ such that $V \oplus W = M \times \mathbb{C}^N$. Check the operator

$$D_V : C_c^\infty(M, S \otimes V) \to C_c^\infty(M, S \otimes V)$$

$$D_V = P \begin{bmatrix} D & & \\ & \ddots & \\ & & D \end{bmatrix}_{N\times N} P,$$

where $P : C_c^\infty(M, S \otimes (M \times \mathbb{C}^N)) \to C_c^\infty(M, S \otimes V)$ is the projection, is a Dirac type one. This is called a twisted Dirac operator.

Remark

- Any Dirac type operator is locally a "minimal" Dirac operator twisted by a vector bundle, observing that $2^{\lfloor \frac{\dim M}{2} \rfloor} | \dim S$.
- If $\dim S = 2^{\lfloor \frac{\dim M}{2} \rfloor}$, D is a canonical Dirac operator on M.

Definition

The Clifford algebra $Cl_n := CL(\mathbb{R}^n)$ is the algebra generated by $c_i := c(e_i), 1 \leq i \leq n$ subject to

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

On $\mathbb{R}^n$, the canonical Dirac operator D appears to be

$$D = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} : C_c^\infty(\mathbb{R}^n, S) \to C_c^\infty(\mathbb{R}^n, S)$$

satisfying $D^2 = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$. In fact,

$$Cl_{2k} \cong M_{2^k}(\mathbb{C}) \qquad Cl_{2k+1} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}).$$

Thus, $S = \mathbb{R}^n \times \mathbb{C}^{2^k} \to \mathbb{R}^n$ where $k = \lfloor \frac{\dim M}{2} \rfloor$.

- A canonical Dirac operator may not exist for a general manifold.

Spin(^{c}) structure

- $\pi_1(SO(n)) = \mathbb{Z}_2$ when $n \geq 3$ and $\pi_1(SO(2)) = \mathbb{Z}$;
- Let Spin(n) be the double cover of $SO(n)$;
- Spin$^c(n)$=Spin$(n) \times_{\mathbb{Z}_2} U(1)$.

Definition

An oriented Riemannian manifold M admits a spin(^{c}) structure if the transition functions $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$ have a lift

$$\tilde{\rho}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(^c)$$

satisfying $\tilde{\rho}_{\alpha\beta}\tilde{\rho}_{\beta\gamma}\tilde{\rho}_{\gamma\alpha} = 1$.

- If M is a complex manifold, then M has a spin^c-structure.
- Spin^c is a condition stronger than orientability and known as the K-orientation, i.e., when M has Spin^c structure, one has

$$K^*(M) = K_{n-*}(M)$$

the K-theoretic Poincaré duality.

Proposition

If M has a spin^c structure of even dimension, there is a spinor bundle *$S \to M$ with $\dim S = 2^{\lfloor \frac{\dim M}{2} \rfloor}$ so that the Clifford multiplication of $e \in TM$ on S gives rise to an isomorphism*

$$c : Cl(TM) \to \mathrm{End}(S).$$

A spinor bundle is a Clifford module.

Definition

$S \to M$ is a Clifford module if it has a hermitian metric and has a connection ∇^S compatible with the Levi-Civita connection ∇^{TM}:

$$\langle c(v)s_1, s_2\rangle + \langle s_1, c(v)s_2\rangle = 0 \qquad v \in TM, s_i \in C_c^\infty(M, S)$$

$$\nabla_X^S(Ys) = (\nabla_X^{TM} Y)s + Y\nabla_X^S s \qquad X, Y \in C_c^\infty(M, TM), s \in C_c^\infty(M, S).$$

Canonical Dirac operators

Definition

Let M be a Riemannian manifold with a spin$(^c)$ structure and S be the spinor bundle. The Dirac operator is given by

$$D = \sum_{i=1}^{n} c(e_i)\nabla^S_{e_i} : C_c^\infty(M, S) \to C_c^\infty(M, S)$$

where $\{e_i\}$ is a local orthonormal basis for TM.

- D is formally sefladjoint, first order differential operator satisfying
$$c(df)^2 = -\|df\|^2.$$
- If M has even dimension, then $S = S^+ \oplus S^-$ and D has the form
$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \qquad (D^+)^* = D^-.$$

Canonical Dirac operators: Examples

- If M has a spin^c structure, the spinor bundle $S \to M$ satisfies

$$\Lambda^* M \cong Cl(TM) \cong \mathrm{End}(S) \cong S \otimes S^*$$

  and a canonical Dirac operator

$$D : C_c^\infty(M, S) \to C_c^\infty(M, S)$$

  such that $d + d^* = D_S$.
- If M is a complex manifold, then M has a spin^c-structure and $S = \Lambda^{0,*} M$ is the spinor bundle and the Dolbeault operator $\bar\partial + \bar\partial^*$ is the canonical Dirac operator.

2. Representations and Dirac Operators

Dirac Operators in Representation Theory

Dirac operators: a geometric tool in representation theory.

- Borel-Weil-Bott: a geometric realization a connected compact Lie group G using twisted Dirac operators over G/T (T : maximal torus).
- Harish-Chandra: classification of discrete series representations of semisimple Lie groups.
- Parthasarathy and Atiyah-Schmid: Used L^2-kernels of twisted Dirac operators on G/K to construct discrete series (K : maximal compact subgroup).
- Vogan: algebraic construction by introducing Dirac cohomology.
- Dirac element in Connes-Kasparov isomorphism, a formula of computing the K-theory of the reduced group C^*-algebra of G (proved using different methods in different contexts by Wassermann, Lafforgue, Chabert-Echterhoff-Nest, Clare-Crisp-Higson, etc).
- . . . .

- Let G be a real or complex semisimple Lie group and K a maximal compact subgroup.
- Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition.
- $\mathfrak{p}$ can be identified with $V = T_{x_0}(K\backslash G)$, $(x_0 = Ke \in K\backslash G)$.
- The killing form on $\mathfrak{g}$ induces a K-invariant inner product on $\mathfrak{p}$

The manifold $M = K\backslash G$ is a Riemmanian manifold with a G-invariant metric.

Definition

$M = K\backslash G$ has a G-equivariant spin structure if $Ad : K \to SO(V)$ can be lifted to

$$K \to Spin(V).$$

Let $G = SL(2,\mathbb{R})$ and $K = SO(2)$. Then

$$M = K\backslash G = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}.$$

Let $\tilde{G}, \tilde{K}$ be the double cover of G, K, then $M = \tilde{K}\backslash\tilde{G}$ has a $\tilde{G}$-equivariant spin structure.

- Assume a G-equivariant spin structure of $K\backslash G$.
- Denote by Δ the irreducible representation of $Spin(\mathfrak{p})$.
- When V has even dimension, one has an isomorphism

$$c : Cl(\mathfrak{p}) \to End(\Delta).$$

Definition

The Dirac operator $D : (C_c^\infty(G) \otimes \Delta)^K \to (C_c^\infty(G) \otimes \Delta)^K$ is given by

$$D = \sum_{i=1}^{n} c(p_i)p_i \qquad \{p_i\} \text{ o.n. basis for } \mathfrak{p},$$

where $(Pf)(g) := \frac{d}{dt}|_{t=0} f(e^{-tP}g)$ for $P \in \mathfrak{p}, g \in G, f \in C_c^\infty(G)$.

- $(C_c^\infty(G) \otimes \Delta)^K$ is the section of the induced representation $\mathrm{Ind}_K^G \Delta$ and identified with $C_c^\infty(K\backslash G, G \times_K \Delta)$;
- D is the Dirac operator on M with spinor bundle

$$S = G \times_K \Delta.$$

Square of twisted Dirac operators

Let G be a real or complex semisimple Lie group.

- Let $V = V_\tau$ be an irreducible unitary representation of K with highest weight τ;
- τ belongs to the closed Weyl chamber ($i\mathfrak{t}^*$ by the Weyl group action) and $\mathfrak{t}$ is the Lie algebra of the maximal torus T in K;
- $\Omega_G = -\sum_{j=1}^{\dim K} X_j^2 + \sum_{i=1}^{\dim(K\backslash G)} Y_i^2$ is the Casimir operator on G where X_j, Y_i are o.n. basis for $\mathfrak{k}, \mathfrak{p}$ resp.

Let $D_V : (C_c^\infty(G) \otimes \Delta \otimes V)^K \to D_V : (C_c^\infty(G) \otimes \Delta \otimes V)^K$ be the Dirac operator on $K\backslash G$ twisted by V.

Theorem (Parthasarathy)

$$D_V^2 = -\Omega_G + \langle \tau + 2\rho_K, \tau \rangle - \langle \rho, \rho \rangle + \langle \rho_K, \rho_K \rangle$$

where

- ρ_K *is the half sum of positive roots of* K *with respect to* T*;*
- ρ *is the half sum of positive roots of* G *with respect to* T*.*

3. Generalized Fredholm index for Dirac Operators

Dirac Operators: Propoerties

Let M be a complete Riemannian manifold and D a Dirac type operator. Then D is

- essentially selfadjoint;
- an elliptic operator (its principal symbol $\sigma_D = ic(\xi)$ is invertible $\forall \xi \neq 0$).

Assume M to be a closed even dimensional manifold, where an elliptic operator is Fredholm.

Definition

The Fredholm index of D is defined by

$$\mathrm{ind}(D) = \dim\ker D^+ - \dim\ker D^-.$$

Applying functional calculus, one has a map

$$C_0(\mathbb{R}) \to \mathcal{K}(L^2(M,S)) \qquad f \mapsto f(D).$$

which extends to the multipliers

$$C_b(\mathbb{R}) \to \mathcal{B}(L^2(M,S)) \qquad f \mapsto f(D).$$

K-theoretic index and Fredholm index

Choose an odd increasing function $\chi : \mathbb{R} \to (-1, 1)$ such that $\lim_{x\to\infty} \chi(x) = 1$. Then $\chi^2 - 1 \in C_0(\mathbb{R})$. Thus

$$\chi(D) \in \mathcal{B}(L^2(M, S)) \quad \text{and} \quad \chi(D)^2 - I \in \mathcal{K}(L^2(M, S)).$$

Definition

The *K-theoretic index* is defined by

$$\mathrm{ind}(D) = \partial[\chi(D)]$$

where

$$\partial : K_1(\mathcal{B}(L^2(M, S))/\mathcal{K}(L^2(M, S))) \to K_0(\mathcal{K}(L^2(M, S))) \cong \mathbb{Z}$$

is the boundary map of the six-term exact sequence associated to

$$0 \to \mathcal{K}(L^2(M, S)) \to \mathcal{B}(L^2(M, S)) \to \mathcal{B}(L^2(M, S))/\mathcal{K}(L^2(M, S)) \to 0.$$

Remark

The Fredholm index of D is equal to the K-theoretic index.

Noncompact case

When M is a complete Riemannian manifold, a Dirac type operator D is essentially selfadjoint. Hence there is a morphism

$$C_0(\mathbb{R}) \to \mathcal{B}(L^2(M, S)).$$

Note that

- D is not necessarily Fredholm.
- the image does not necessarily lands in compact operators.

When a locally compact group G acts on M properly, cocompactly and isometrically,

- D is a "generalized Fredholm operator" on a Hilbert module over $C_r^*(G)$.

Hilbert $C_r^*(G)$-module

- $C_c^\infty(M)$ carries a $C_c^\infty(G)$-valued inner product $\langle\cdot,\cdot\rangle_G$:

$$\langle\xi,\eta\rangle_G : g \mapsto \int_M \langle\xi(x),\eta(xg)\rangle dx \quad \xi,\eta \in C_c^\infty(M);$$

- $C_c^\infty(M)$ is also a $C_c^\infty(G)$-module:

$$(\xi * f)(x) = \int_G \xi(xg^{-1})f(g)dg \qquad x \in M, \xi \in C_c^\infty(M), f \in C_c^\infty(G).$$

- Thus, for $\xi,\eta \in C_c^\infty(M)$ and $f \in C_c^\infty(G)$ one has

$$\langle\xi,\eta * f\rangle_G = \langle\xi,\eta\rangle_G * f.$$

- Complete $C_c^\infty(M)$ under $\langle\cdot,\cdot\rangle_G$ to obtain $\mathcal{E}_G(M)$, a Hilbert module over $C_r^*(G)$.
- Complete $C_c^\infty(M,S)$ to obtain $\mathcal{E}_G(M,S)$.

Example

When $M = K\backslash G$ and $S = G \times_K \Delta$: $\mathcal{E}_G(M,S) = (L^2(G)\otimes\Delta)^K$

K-theoretic Index

- D is a regular essentially selfadjoint operator on $\mathcal{E}_G(M,S)$.
- When M/G is compact, one has

$$C_0(\mathbb{R}) \to \mathcal{K}(\mathcal{E}_G(M,S)) \qquad f \mapsto f(D).$$

- Let $\chi : \mathbb{R} \to (-1,1)$ be an odd increasing function such that $\lim_{x\to\infty} \chi(x) = 1$. Then $\chi(D)^2 - I \in \mathcal{K}(\mathcal{E}_G(M,S))$.

Definition

The higher index or the equivariant index of D (odd $\mathbb{Z}_2$-graded) is defined by

$$\mathrm{ind}_G(D) := \partial[\chi(D)] \in K_0(\mathcal{K}(\mathcal{E}_G(M,S))) \cong K_0(C_r^*(G)).$$

Here $\partial : K_1(\mathcal{B}(\mathcal{E}_G(M,S))/\mathcal{K}(\mathcal{E}_G(M,S))) \to K_0(\mathcal{K}(\mathcal{E}_G(M,S)))$ is the boundary map of the K-theoretic six-term exact sequence associated to the short exact sequence

$$0 \to \mathcal{K}(\mathcal{E}_G(M,S)) \to \mathcal{B}(\mathcal{E}_G(M,S)) \to \mathcal{B}(\mathcal{E}_G(M,S))/\mathcal{K}(\mathcal{E}_G(M,S)) \to 0.$$

4. Dirac Induction

Homogeneous vector bundle

- Let G be an almost connected Lie group and K is a maximal compact subgroup.
- Associated to a representation (ρ, V) of K, one can construct the homogeneous vector bundle

$$\mathcal{V} := K \backslash (V \times G)$$

where the action is given by

$$k(v, g) = (\rho(k)v, k^{-1}g) \qquad k \in K, v \in V, g \in G.$$

- Let $R(K)$ the representation ring generated by irreducible unitary representations of K. Then

$$R(K) \to K^0_G(K \backslash G) \qquad [V] \mapsto [\mathcal{V}].$$

Dirac Induction

- Assume $K\backslash G$ has a G-equivariant spin structure;
- D is the canonical Dirac operator on the spinor bundle $S = G \times_K \Delta$.
- Form the twisted Dirac operator

$$D_V = D_{\mathcal{V}} : C_c^\infty(K\backslash G, S \otimes \mathcal{V}) \to C_c^\infty(K\backslash G, S \otimes \mathcal{V}).$$

Definition

The Dirac induction is a map

$$R(K) \to K_i(C_r^*(G)) \qquad [V] \mapsto \mathrm{ind}_G(D_V) \in K_n(C_r^*(G))$$

Here n is the parity of $\dim(K\backslash G)$.

Remark

If $K\backslash G$ has no G-equivariant spin structure, replace $R(K)$ by

$$R^1(K) = \{\pi \in R(\tilde{K}) \text{ that factors through } K\}.$$

Connes-Kasparov isomorphism

Theorem (Penington-Plymen, Wassermann, Valette, Lafforgue, Chabert-Echterhoff-Nest, Clare-Crisp-Higson, etc)

The Dirac induction map

$$R(K) \to K_n(C_r^*(G)) \qquad [V] \mapsto \mathrm{ind}_G(D_V)$$

is an isomorphism of abelian groups ($n = \dim(K \backslash G)$).

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