Stochastic quantization of the Φ^3_3-model

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Euclidean Φ^k_d field theory

Defocusing Φ^k_d-measures on $\mathbb{T}^d$:

$$d\rho(u) = Z^{-1} \exp\Big(- \frac{1}{k} \int_{\mathbb{T}^d} u^k dx \Big) d\mu(u),$$

associated with the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{T}^d} |\langle \nabla \rangle u|^2 dx + \frac{1}{k} \int_{\mathbb{T}^d} u^k dx$$

where $\langle \nabla \rangle = \sqrt{1 - \Delta}$

- $\mu = \mu_d$ is the massive Gaussian free field on $\mathbb{T}^d$
- $k \geq 4$ is an even integer

The construction and various properties of the (defocusing) Φ^k_d-measures have been studied extensively (60's $\sim$) for (i) $d = 2$, $k \in 2\mathbb{N} + 2$ and (ii) $d = 3$, $k = 4$:

- Nelson, Glimm, Jaffe, Feldman, Park, Brydges-Fröhlich-Sokal, ...
- More recently: Albeverio-Kusuoka, Gubinelli-Hofmanová, Barashkov-Gubinelli
- $(d, k) = (4, 4)$: Φ^4_4-model is critical (triviality by Aizenman and Duminil-Copin '21)

Also, studied: (i) on $\mathbb{R}^d$ and (ii) associated dynamical problems (stochastic quantization)

Focusing Gibbs measures

Focusing Φ^k_d-measures on $\mathbb{T}^d$:

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{k}\int_{\mathbb{T}^d} u^k dx\right) d\mu(u)$$

- By "focusing", we mean ***non-defocusing***, namely

  $\sigma > 0$ when $k \in 2\mathbb{N}+2$ and $\sigma \in \mathbb{R}\setminus\{0\}$ when $k \in 2\mathbb{N}+1$

  so that the interaction potential $\frac{\sigma}{k}\int_{\mathbb{T}^d} u^k dx$ is *unbounded from above*

  (When $k \in 2\mathbb{N}+1$, the sign of σ does not play any role)

$d = 1$: Lebowitz-Rose-Speer '88 studied the construction of the Gibbs measures

$$d\rho(u) = Z^{-1} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) d\mu(u), \quad p > 2,$$

associated with the *focusing* nonlinear Schrödinger equations (NLS) on $\mathbb{T}$:

$$i\partial_t u = \partial_x^2 u + |u|^{p-2} u$$

(u = complex-valued)

Focusing Gibbs measure on $\mathbb{T}$:

$$d\rho(u) = Z^{-1} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) d\mu(u)$$

is ***not*** normalizable for $p > 2$

Lebowitz-Rose-Speer '88 proposed to consider the following two options:

(i) with an L^2-cutoff:

$$d\rho(u) = Z^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} |u|^2 dx \le K\}} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) d\mu(u)$$

(ii) **generalized grand-canonical Gibbs measure:** with a taming by the L^2-norm,

$$d\rho(u) = Z^{-1} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx - A\left(\int_{\mathbb{T}} u^2 dx\right)^{\gamma}\right) d\mu(u)$$

for some appropriate $\gamma = \gamma(p) > 0$,

- $A =$ (generalized) chemical potential)
- Carlen-Fröhlich-Lebowitz '16

Focusing Gibbs measure on $\mathbb{T}$ with an L^2-cutoff:

- $2 < p < 6$ (subcritical): normalizable for any $K > 0$ (Lebowitz-Rose-Speer '88, Bourgain '94)
- $p > 6$ (supercritical): non-normalizable for any $K > 0$ (Lebowitz-Rose-Speer '88)
- Also applies to the GGC formulation (Carlen-Fröhlich-Lebowitz '16)

  Remark: For any $K > 0$, $\gamma > 0$, and $A > 0$,

$$\mathbf{1}_{\{|\cdot| \le K\}}(x) \le \exp\big(-A|x|^\gamma\big)\exp\big(AK^\gamma\big)$$

  $\Longrightarrow Z_{L^2\text{-cutoff}} \lesssim Z_{\text{GGC}}$

Critical case $p = 6$: Let Q be the unique optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$:

$$\|u\|_{L^6(\mathbb{R})}^6 \le C_{\text{GNS}} \|u'\|_{L^2(\mathbb{R})}^2 \|u\|_{L^2(\mathbb{R})}^4$$

such that $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$. The following phase transition holds:

- non-normalizable if $K > \|Q\|_{L^2(\mathbb{R})}^2$ (Lebowitz-Rose-Speer '88)
- normalizable if $K < \|Q\|_{L^2(\mathbb{R})}^2$ (Bourgain '94 for $K \ll 1$, Oh-Sosoe-Tolomeo '21)

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Phase transition at the critical mass $K = \|Q\|^2_{L^2(\mathbb{R})}$ when $p = 6$
$\Longrightarrow$ focusing Φ^6_1-model is *critical*

Furthermore, at the *critical* mass threshold $K = \|Q\|^2_{L^2(\mathbb{R})}$:

- minimal mass finite-time blowup solution to the focusing NLS (with $\int_{\mathbb{T}} |u(t)|^6 dx \to \infty$ as $t \to T_{\text{blowup}}$): Ogawa-Tsutsumi '90

but somewhat unexpectedly,

- focusing Φ^6_1-measure is ***normalizable*** at $K = \|Q\|^2_{L^2(\mathbb{R})}$: Oh-Sosoe-Tolomeo '21
(in particular, we do not see the minimal mass blowup solutions probabilistically)

Hence, the construction of the 1-d focusing Gibbs measures is complete

$d = 2$: A typical element u under the GFF μ is not a function

$\Longrightarrow$ (Wick) renormalization is needed

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{|\int_{\mathbb{T}^2}:u^2:\,dx|\le K\}}\exp\left(\frac{1}{k}\int_{\mathbb{T}^2}:u^k:\,dx\right)d\mu(u)$$

$k = 4$:

- L^2-critical (as in $(d,p) = (1,6)$)
- focusing Φ^4_2-measure is *not* normalizable (for any $K > 0$): Brydges-Slade '96, Oh-Seong-Tolomeo '20 (also for the mass-critical quartic case on $\mathbb{T}^d$ for any $d \ge 1$)

$k = 3$: (real-valued)

- normalizable: Jaffe, mid 90's but not suitable for heat/wave/Schrödinger dynamics
- Bourgain '95: generalized grand-canonical formulation of the Φ^3_2-measure:

$$d\rho(u) = Z^{-1}e^{\frac{\sigma}{3}\int_{\mathbb{T}^2}:u^3:\,dx - A\left(\int_{\mathbb{T}^2}:u^2:\,dx\right)^2}d\mu(u)$$

  for sufficiently large $A > 0$. Also, Oh-Seong-Tolomeo '20 for any $\sigma \neq 0$

  - Associated (stochastic) nonlinear wave (NLW) dynamics: Oh-Thomann '20, Gubinelli-Koch-Oh '18, GKO-Tolomeo '20

There is *no* critical case in the 2-d case

Critical Φ_3^3-measure

$d = 3$: $k = 3$ is the smallest power

$\Longrightarrow$ **Φ_3^3-measure:** Oh-Okamoto-Tolomeo '21

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\left|\int_{\mathbb{T}^3} :u^2:\, dx\right|^\gamma\right) d\mu(u)$$

- turns out to be *critical*:
  - normalizability in the weakly nonlinear regime ($0 < |\sigma| \ll 1$)
  - non-normalizability in the strongly nonlinear regime ($|\sigma| \gg 1$)
- The Φ_3^3-measure ρ and GFF μ are mutually singular (just like the defocusing Φ_3^4-measure)
- The Φ_3^3-model is L^2-subcritical (unlike the $d = 1, 2$ case)

  $\Longrightarrow$ non-normalizability is *not* related to finite-time blowup solutions
- real-valued (i.e. not suitable for studying NLS dynamics)

Basic setup:

- $\pi_N =$ "suitable" frequency projection onto $\{|n| \lesssim N\}$ (more later)
- Given $\text{Law}(u) = \mu$, let $u_N = \pi_N u$. Then, For each fixed $x \in \mathbb{T}^3$, $u_N(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\big[u_N^2(x)\big] \sim \sum_{|n| \le N} \frac{1}{\langle n \rangle^2} \sim N \longrightarrow \infty, \quad \text{as } N \to \infty$$

- Define the Wick powers $:u_N^2:$ and $:u_N^3:$ by setting

$$:u_N^2: = H_2(u_N; \sigma_N) = u_N^2 - \sigma_N \quad \text{and} \quad :u_N^3: = H_3(u_N; \sigma_N) = u_N^3 - 3\sigma_N u_N$$

- **Renormalized potential energy:**

$$R_N^\diamond(u) = -\frac{\sigma}{3} \int_{\mathbb{T}^3} :u_N^3: dx + A \bigg| \int_{\mathbb{T}^3} :u_N^2: dx \bigg|^\gamma + \alpha_N$$

for a suitable diverging constant $\alpha_N \to \infty$, providing a further renormalization

Weakly nonlinear regime: $0 < |\sigma| \ll 1$

- **Truncated Φ^3_3-measure:** $d\rho_N(u) = Z_N^{-1} e^{-R_N^\diamond(u)} d\mu(u)$

Theorem 1.a: Oh-Okamoto-Tolomeo '21

Let $0 < |\sigma| \ll 1$. Then, by choosing $\gamma = 3$ and $A = A(\sigma) \gg 1$, we have

1. uniform exponential integrability of the density:

$$\sup_{N\in\mathbb{N}} Z_N = \sup_{N\in\mathbb{N}} \Big\| e^{-R_N^\diamond(u)} \Big\|_{L^1(\mu)} < \infty$$

2. the truncated Φ^3_3-measure ρ_N converges weakly to a *unique* limit ρ, formally given by

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3} \int_{\mathbb{T}^3} :u^3:\, dx - A \left| \int_{\mathbb{T}^3} :u^2:\, dx \right|^3 - \infty \right) d\mu(u)$$

3. the limiting Φ^3_3-measure ρ and the GFF μ are *mutually singular*
4. the Φ^3_3-measure ρ is absolutely continuous w.r.t. a “shifted measure”

- Analogous to the situation for the defocusing Φ^4_3-measure (in particular, ③ and ④)
- $R_N^\diamond(u)$ does not converge. Neither does the truncated density $e^{-R_N^\diamond(u)}$

Strongly nonlinear regime: $|\sigma| \gg 1$

Theorem 1.b: Oh-Okamoto-Tolomeo '21

Let $|\sigma| \gg 1$, $\gamma \geq 3$, and $A > 0$.

5. **Non-normalizability**: Fix $\delta > 0$. Given $N \in \mathbb{N}$, define the following *tamed version of the truncated Φ^3_3-measure*:

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\Big(-\delta \|\pi_N u\|_{\mathcal{A}}^{20} - R_N^{\diamond}(u)\Big) d\mu(u).$$

Then, $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ converges weakly to some limiting probability measure ν_δ, allowing us to define the following **σ-finite version of the Φ^3_3-measure** on $\mathcal{C}^{-100}(\mathbb{T}^3)$:

$$d\overline{\rho}_\delta = e^{\delta \|u\|_{\mathcal{A}}^{20}} d\nu_\delta = e^{\delta \|u\|_{\mathcal{A}}^{20}} \cdot \lim_{N\to\infty} Z_{N,\delta}^{-1} e^{-\delta \|\pi_N u\|_{\mathcal{A}}^{20} - R_N^{\diamond}(u)} d\mu(u)$$

Furthermore, $\overline{\rho}_\delta$ is not normalizable: $\displaystyle\int 1 \, d\overline{\rho}_\delta = \infty$

- $\mathcal{A} \sim B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$
- We also have $d\overline{\rho}_\delta = e^{\delta\|u\|_{\mathcal{A}}^{20}} \cdot \lim_{N\to\infty} \widehat{Z}_{N,\delta}^{-1} e^{-\delta\|u\|_{\mathcal{A}}^{20}} e^{-R_N^{\diamond}(u)} d\mu(u)$
  ($u \mapsto \|u\|_{\mathcal{A}}$ is not continuous on $\mathcal{C}^{-100}(\mathbb{T}^3)$)

Theorem 1.b: continued

6. **Non-convergence:** The truncated Φ^3_3-measures ρ_N do *not* converge to any weak limit, even up to a subsequence, as probability measures on $\mathcal{A}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$

- support of of GFF $\mu \subset \mathcal{C}^{-\frac{1}{2}-}(\mathbb{T}^3)$

  $\Longrightarrow$ No weak convergence of $\{\rho_N\}_{N\in\mathbb{N}}$ as measures on a "natural" space

  (still possible to converge as measures in a very weak topology, in a pathological manner)

Main ingredients (for both weakly and strongly nonlinear regimes)**:**

- Variational approach as in Barashkov-Gubinelli '20
- A novel approach to prove non-normalizability, based on / inspired by Tolomeo-Weber '21, Oh-Okamoto-Tolomeo '20, Oh-Seong-Tolomeo '20

On the frequency projector

Given $N \in \mathbb{N}$, our frequency projector π_N is defined by

$$\pi_N f = \pi_N^{\text{cube}} f = \sum_{n \in \mathbb{Z}^3} \mathbf{1}_Q(N^{-1}n) \widehat{f}(n) e_n, \qquad Q = [-1,1]^3.$$

Namely, π_N is a projection onto $\{n = (n_1, n_2, n_3) \in \mathbb{Z}^3 : \max_{j=1,2,3} |n_j| \le N\}$

(i) π_N is bounded on $L^p(\mathbb{T}^3)$ for any $1 < p < \infty$

(ii) π_N is a projection, i.e. orthogonality $(\mathrm{Id} - \pi_N)\pi_N = 0$ (very important)

Summary of Theorems 1.a & 1.b:

1. uniform integrability and tightness
2. uniqueness of the limiting Φ_3^3-measure $\Longleftarrow$ (ii)
3. mutual singularity
4. non-normalizability
5. non-convergence

- Property (i) is needed for ⑤ (For example, the projection π_N^{ball} onto $\{|n| \le N\}$ does not work)
- Property (ii) is needed for ②, ④, and ⑤ (smooth projection or mollification does not work)

Focusing Hartree Gibbs measure

Hartree Φ^4_3-measure on $\mathbb{T}^3$: $d\rho = Z^{-1} \exp\left(\frac{\sigma}{4} \int_{\mathbb{T}^3} (V * u^2) u^2 \, dx \right) d\mu$

- V = kernel for Bessel potential $\langle \nabla \rangle^{-\beta}$ of order $\beta > 0$: $\widehat{V}(n) = \langle n \rangle^{-\beta}$

  $\Longrightarrow$ ($\beta = 2$: Coulomb potential $\sim \Phi^3_3$-model)

Focusing case ($\sigma > 0$): proposed by Lebowitz 90's

either (a) with an Wick-ordered L^2-cutoff or (ii) in the generalized grand-canonical formulation

Theorem 2: Oh-Okamoto-Tolomeo '20

(i) $\beta > 2$ (subcritical): the Gibbs measure ρ exists (also, Bourgain '97 for (a))

(ii) $\beta < 2$ (supercritical): ρ is not normalizable

(iii) critical case $\beta = 2$: The following phase transition holds:

- weakly nonlinear regime ($0 < \sigma \ll 1$): ρ exists
- strongly nonlinear regime ($\sigma \gg 1$): ρ is not normalizable

The focusing Hartree Φ^4_3-model is critical when $\beta = 2$ but is simpler than the Φ^3_3-model

- Truncated densities converge in $L^p(\mu)$ for any $1 < p < \infty$

  $\Longrightarrow$ ρ_N converges to ρ in total variation and $\rho \ll \mu$

- Also, the non-normalizability part is more straightforward: $\sup_{N \in \mathbb{N}} Z_N = \infty$

A few words on the proof

Variational approach (after Barashkov-Gubinelli '20)**:**

- $\text{Law}(Y) = \mu$ (We choose $Y(t) = \langle \nabla \rangle^{-1} W(t)$ and set $Y = Y(1)$)
- Drifts: $\mathbb{H}_a^s = \big\{\theta \in L^2([0,1]; H^s(\mathbb{T}^3)),\ \mathbb{P}\text{-almost surely, progressively measurable}\big\}$
- **Boué-Dupuis variational formula:** B-D '98, Üstünel '14

  Fix $N \in \mathbb{N}$. For "nice" functions $F : C^\infty(\mathbb{T}^3) \to \mathbb{R}$,

$$-\log \mathbb{E}\Big[e^{-F(Y_N)}\Big] = \inf_{\theta \in \mathbb{H}_a^0} \mathbb{E}\bigg[F(Y_N + \Theta_N(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 \, dt\bigg],$$

  where $Y_N = \pi_N Y,\ \Theta_N = \pi_N \Theta,$ and $\Theta(t) = \int_0^t \langle \nabla \rangle^{-1} \theta(t') dt'$

  $\Longleftarrow$ useful in bounding a partition function <u>both from above and from below</u>

Change of variables:

$$-\frac{\sigma}{3}\int_{\mathbb{T}^3} :(Y_N+\Theta_N)^3: dx = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y_N^3: dx - \sigma \boxed{\int_{\mathbb{T}^3} :Y_N^2: \Theta_N dx}$$
$$- \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta_N^3 dx.$$

- Define $\mathfrak{Z}^N$ by with $\mathfrak{Z}^N(0)=0$ by $\boxed{\dot{\mathfrak{Z}}^N(t) = (1-\Delta)^{-1} :Y_N^2(t):}$ $\sim 1-$
- A change of variables: $\boxed{\dot{\Upsilon}^N(t) = \dot{\Theta}(t) - \sigma \dot{\mathfrak{Z}}_N(t)}$, where $\mathfrak{Z}_N = \pi_N \mathfrak{Z}^N$

$\Longrightarrow$ Then, by Ito's formula, we have

$$\mathbb{E}\Big[-\sigma\int_{\mathbb{T}^3} :Y_N^2: \Theta_N dx + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\Big] = \frac{1}{2}\mathbb{E}\Big[\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big] - \alpha_N,$$

where $\alpha_N = \frac{\sigma^2}{2}\mathbb{E}\big[\int_0^1 \|\dot{\mathfrak{Z}}_N(t)\|_{H_x^1}^2 dt\big] \longrightarrow \infty$ as $N \to \infty$

$\Longrightarrow$ yields the second renormalization in

$$R_N^{\diamond}(u) = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :u_N^3: dx + A\Big|\int_{\mathbb{T}^3} :u_N^2: dx\Big|^{\gamma} + \alpha_N$$

1. Uniform exponential integrability and tightness:

- View $\dot{\Upsilon}^N \in \mathbb{H}_a^1$ as a drift and study the minimization problem:

$$-\log Z_N = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[R_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{L_x^2}^2 \, dt\Big]$$

Goal: Show that RHS is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$ and $\dot{\Upsilon}^N \in \mathbb{H}_a^1$

- pathwise analysis and use pathwise bounds on Y_N and $\mathfrak{Z}_N$. Some terms are critical in terms of scaling. No room to spare, even logarithmically
- Use the positive part $\mathcal{U}_N$ to hide contributions with Υ_N:

$$\mathcal{U}_N(\dot{\Upsilon}^N) = \mathbb{E}\Big[\frac{A}{2}\Big|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\Big|^3 + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 \, dt\Big]$$

2. Uniqueness of the limiting Φ^3_3-measure:

- Take two weakly convergent subsequences $\{\rho_{N_k^1}\}_{k=1}^\infty$ and $\{\rho_{N_k^2}\}_{k=1}^\infty$. By the variational approach, show

$$\lim_{k\to\infty}\int \exp(F(u))d\rho_{N_k^1} \geq \lim_{k\to\infty}\int \exp(F(u))d\rho_{N_k^2}$$

for any bounded Lipschitz continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$

- crucially replies on the orthogonality $(\mathrm{Id} - \pi_{N_k^2})\pi_{N_k^2} = 0$ for

$$\int_{\mathbb{T}^3} \big((\mathrm{Id} - \pi_{N_k^2})Y\big)\big(\pi_{N_k^2}\underline{\Upsilon}^{N_k^2}\big)dx = 0$$

$\Longleftarrow$ critical in terms of scaling with no room to spare, not even logarithmically

3. Singularity of the limiting Φ^3_3-measure:

- We show that there exists $N_k \to \infty$ such that the set

$$\Sigma = \Big\{u \in H^{-\frac{1}{2}-}(\mathbb{T}^3) : \lim_{k\to\infty}(\log N_k)^{-\frac{3}{4}}(R^\diamond_{N_k}(u) - \alpha_{N_k}) = 0\Big\}$$

satisfies

$$\mu(\Sigma) = 1 \qquad \text{but} \qquad \rho(\Sigma) = 0$$

without using the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$
(Barashkov-Gubinelli '20 uses the shifted measure)

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for any bounded Lipschitz continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$

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$$\int_{\mathbb{T}^3} \big((\mathrm{Id} - \pi_{N^2_k})Y\big)\big(\pi_{N^2_k}\underline{\Upsilon}^{N^2_k}\big)dx = 0$$

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without using the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$
(Barashkov-Gubinelli '20 uses the shifted measure)

4. Non-normalizability: $\|u\|_{\mathcal{A}} = \sup_{0<t\leq 1} \big(t^{\frac{3}{8}} \|e^{t\Delta}u\|_{L^3(\mathbb{T}^3)}\big)$

1. First, construct a σ-finite version $\bar{\rho}_\delta$ of the Φ^3_3-measure ($\mathcal{A} \sim B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3)$):

$$d\bar{\rho}_\delta = e^{\delta\|u\|^{20}_{\mathcal{A}}} d\nu_\delta = e^{\delta\|u\|^{20}_{\mathcal{A}}} \cdot \lim_{N\to\infty} Z^{-1}_{N,\delta}\, e^{-\delta\|\pi_N u\|^{20}_{\mathcal{A}} - R^{\diamond}_N(u)} d\mu$$

2. Need to construct drifts, achieving the desired divergence in the variational formulation:

"drift $\sim -Y$ + a perturbation"

with the perturbation is bounded in $L^2(\mathbb{T}^3)$ but has a large L^3-norm, thus having a highly concentrated profile, such as a soliton or a finite time blowup profile (Tolomeo-Weber '21, Oh-Okamoto-Tolomeo '20, Oh-Seong-Tolomeo '20)

5. Non-convergence as measures on $\mathcal{A}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$:

- Suppose by contradiction that, as probability measures on $\mathcal{A}$, $\{\rho_{N_k}\}_{k\in\mathbb{N}}$ has a weak limit ν_0. Then, with

$$1 = \int 1\, d\nu_0 = \int \exp\Big(-\delta\|u\|^{20}_{\mathcal{A}}\Big) d\nu_0(u) \int 1\, d\bar{\rho}_\delta(u) = \infty$$

$\Longrightarrow$ contradiction

Dynamical problem

Stochastic quantization: Parisi-Wu '81, Ryang-Saito-Shigemoto '85

Study an SPDE which preserves the target measure (= Φ^3_3-measure in our case)

Parabolic Φ^3_3-model: $\partial_t u + (1-\Delta)u - \sigma :u^2: + M(:u^2:)\, u = \sqrt{2}\xi$

- $M(:u^2:) = 6A\left|\int_{\mathbb{T}^3} :u^2: dx\right| \int_{\mathbb{T}^3} :u^2: dx$ $\Longleftarrow$ from the taming in the Φ^3_3-measure
- Well-posedness of the parabolic Φ^3_3-model follows easily from the first order expansion: $u = \,^\bullet\!\!\mid + v$

Hyperbolic Φ^3_3-model: stochastic damped NLW on $\mathbb{T}^3$

$$\partial_t^2 u + \partial_t u + (1-\Delta)u - \sigma :u^2: + M(:u^2:)\, u = \sqrt{2}\xi$$

- canonical stochastic quantization equation for the Φ^3_3-measure (= Hamiltonian SQE, given as a Langevin equation)

Goal: Construct global-in-time dynamics for the hyperbolic Φ^3_3-model and prove invariance of the Φ^3_3-measure

- Let $0 < |\sigma| \ll 1$ (weakly nonlinear regime), $\gamma = 3$, and $A \gg 1$ such that the Φ^3_3-measure ρ exists:

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\bigg|\int_{\mathbb{T}^3} :u^2:\, dx\bigg|^3 - \infty\bigg) d\mu(u)$$

- Define the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$, where μ_0 = (spatial) white noise on $\mathbb{T}^3$

$\Longleftarrow$ formally invariant under the hyperbolic Φ^3_3-model

Theorem 2: Oh-Okamoto-Tolomeo '21

- The hyperbolic Φ^3_3-model on $\mathbb{T}^3$ is almost surely globally well-posed w.r.t. the Gibbs measure $\vec{\rho}$
- the Gibbs measure $\vec{\rho}$ is invariant under the hyperbolic Φ^3_3-dynamics

- A similar result holds for the quadratic NLW on $\mathbb{T}^3$ (with no damping/noise)

Two separate problems:

1. Local well-posedness: paracontrolled approach (Gubinelli-Koch-Oh '18)
2. Almost sure global well-posedness (beyond Bourgain's invariant measure argument)

Paraproduct and paracontrolled distribution

Paraproduct decomposition (Bony '81)**:** Given two functions f and g on $\mathbb{T}^3$ of regularities s_1 and s_2, we write the product fg as

$$fg = f \olessthan g + f \ominus g + f \ogreaterthan g$$

$$\stackrel{\text{def}}{=} \sum_{j<k-1} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{|j-k|\le 1} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{k<j-1} \mathbf{P}_j f \, \mathbf{P}_k g$$

where $\mathbf{P}_j$ = Littlewood-Paley projection onto frequencies $\{n \in \mathbb{Z}^3 : |n| \sim 2^j\}$

- Paraproduct $f \olessthan g$ (of g by f): *always well defined* as a distribution of regularity $\min(s_2, s_1 + s_2)$
  $\Longleftarrow$ at best as smooth as g (high freq piece)
- Resonant product $f \ominus g$: well defined (in general) only if $s_1 + s_2 > 0$
  $\Longleftarrow$ main difficulty

Paracontrolled distribution (Gubinelli-Imkeller-Perkowski '15) **:**

We say f is *paracontrolled* by g if

$$f = f' \olessthan g + R$$

for some f' (basically Gubinelli derivative) and a *smoother* remainder R

Note: This is a way to impose a structure

Paraproduct and paracontrolled distribution

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Local theory via the paracontrolled approach

- Ψ = stochastic convolution with the GFF initial data:

$$\begin{cases} \partial_t^2 \Psi + \partial_t \Psi + (1-\Delta)\Psi = \sqrt{2}\xi \\ (\Psi, \partial_t \Psi)|_{t=0} = (\phi_0, \phi_1) \quad \text{with } \mathrm{Law}(\phi_0, \phi_1) = \mu \otimes \mu_0 \end{cases} \implies \boxed{\Psi \sim -\tfrac{1}{2}-}$$

- Set $\mathcal{I} = (\partial_t^2 + \partial_t - \Delta)^{-1}$ = damped wave Duhamel integral operator
- Second order term $\mathcal{I}(\Psi^{\diamond 2}) = \mathcal{I}(\Psi^{\diamond 2})$, where $\Psi^{\diamond 2}$ is the renormalized Ψ^2.

  If one proceeds with a "parabolic thinking", one expects $0- = 2(-\frac{1}{2}-) + 1$

  In fact, there is an *extra multilinear smoothing* coming from ***dispersion*** in the product structure and we can show that $\mathcal{I}(\Psi^{\diamond 2})$ has regularity $\frac{1}{2}-$ (gain of $\frac{1}{2}$ derivative)

  - multilinear oscillatory sum analysis: Gubinelli-Koch-Oh '18

Second order expansion: $u = \Psi + \mathcal{I}(\Psi^{\diamond 2}) + v$

$$\implies \quad (\partial_t^2 + \partial_t - \Delta)v = 2v\Psi + 2\Psi\,\mathcal{I}(\Psi^{\diamond 2}) + \text{ other terms}$$

- $\Psi\,\mathcal{I}(\Psi^{\diamond 2}) \sim -\frac{1}{2}- \quad \implies \quad v \sim \frac{1}{2}-$

  $\implies$ $v\Psi$ is *not* well defined since $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0.$

We introduce a **paracontrolled ansatz**

Local theory via the paracontrolled approach

- $\uparrow$ = stochastic convolution with the GFF initial data:

$$\begin{cases} \partial_t^2 \uparrow + \partial_t \uparrow + (1-\Delta)\uparrow = \sqrt{2}\xi \\ (\uparrow, \partial_t \uparrow)|_{t=0} = (\phi_0, \phi_1) \quad \text{with } \operatorname{Law}(\phi_0,\phi_1) = \mu \otimes \mu_0 \end{cases} \implies \boxed{\uparrow \sim -\tfrac{1}{2}-}$$

- Set $\mathcal{I} = (\partial_t^2 + \partial_t - \Delta)^{-1}$ = damped wave Duhamel integral operator
- Second order term $\curlyvee = \mathcal{I}(\vee)$, where $\vee$ is the renormalized $\uparrow^2$.

  If one proceeds with a "parabolic thinking", one expects $0- = 2(-\frac{1}{2}-) + 1$

  In fact, there is an *extra multilinear smoothing* coming from ***dispersion*** in the product structure and we can show that $\curlyvee$ has regularity $\frac{1}{2}-$ (gain of $\frac{1}{2}$ derivative)

  - multilinear oscillatory sum analysis: Gubinelli-Koch-Oh '18

Second order expansion: $u = \uparrow + \curlyvee + v$

$$\implies \qquad (\partial_t^2 + \partial_t - \Delta)v = 2v\uparrow + 2\uparrow\curlyvee + \text{ other terms}$$

- $\uparrow\curlyvee \sim -\frac{1}{2}- \quad \implies \quad v \sim \frac{1}{2}-$

  $\implies \quad v\uparrow$ is *not* well defined since $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0$.

We introduce a **paracontrolled ansatz**

Paracontrolled ansatz: Write $u = \uparrow + \curlyvee + X + Y$, where X and Y satisfy

$$
\begin{aligned}
(\partial_t^2 + \partial_t - \Delta) X &= 2(X + Y + \curlyvee) \varolessthan \uparrow - M(:u^2:)\uparrow, \\
(\partial_t^2 + \partial_t - \Delta) Y &= \text{all the other terms}
\end{aligned}
$$

- $:u^2: = (X + Y + \curlyvee)^2 + 2(X+Y)\uparrow + 2\uparrow\curlyvee + \vee$
- X-equation collects the worst terms and is *nonlinear* (unlike GKO '18 and Mourrat-Weber '17 on the parabolic Φ^4_3-model)

Main issue:

- $\uparrow \sim -\frac{1}{2}- \quad \Longrightarrow \quad X$ has regularity $\frac{1}{2}- = (-\frac{1}{2}-) + 1$
  $\Longrightarrow$ the resonant product $X \circleddash \uparrow$ is *not* well-defined
- this problematic term also appears in $M(:u^2:)$ in the X-equation

$\Longrightarrow$ We can *not* proceed as in GKO '18 / MW '17 (where we substitute the Duhamel formulation of the X-equation into the Y-equation)

- This would lead to an infinite iteration of the Duhamel formulation in the X-equation (which could be done)

New idea: introduce a new unknown, representing the problematic resonant product:

$$\text{“}\mathfrak{R} = X \circleddash \<1>\text{”}$$

which leads to a system of ***three unknowns***

- $\boxed{:u^2: = Q_{X,Y} + 2\mathfrak{R} + \<30>^2 + 2\<320> + \<2>}$, $\qquad Q_{X,Y}$ = harmless terms
- Still need to use the *paracontrolled operators* from GKO '18, OOT '20,

$$\mathfrak{I}_{\circledless,\circleddash}(w) = \mathcal{I}(w \circledless \<1>) \circleddash \<1>$$

Hyperbolic Φ^3_3-system for $(X, Y, \mathfrak{R})$**:**

$$(\partial_t^2 + \partial_t + 1 - \Delta)X = 2(X + Y + \<30>) \circledless \<1> - M(:u^2:)\<1>,$$

$$(\partial_t^2 + \partial_t + 1 - \Delta)Y = (X + Y + \<30>)^2 + 2(\mathfrak{R} + Y \circleddash \<1> + \<320>) + 2(X + Y + \<30>) \circledgtr \<1>$$

$$- M(:u^2:)(X + Y + \<30>),$$

$$\mathfrak{R} = 2\mathfrak{I}_{\circledless,\circleddash}(X + Y + \<30>) - \int_0^t M(:u^2:)(t')\mathbb{A}(t,t')dt'$$

- Standard deterministic analysis with the Strichartz estimates yields local well-posedness with a continuous map from an **enhanced data set**

$$\Xi = (\<1>, \<2>, \<30>, \<320>, \mathbb{A}, \mathfrak{I}_{\circledless,\circleddash})$$

to the solution $(X, Y, \mathfrak{R})$

Global-in-time dynamics

Unlike the parabolic Φ^4_3-model,

- No smoothing on linear solutions (no deterministic LWP with the Gibbs initial data)
- The energy functional is not coercive (no deterministic global well-posedness)

In the study of dispersive PDEs, a *conservation law* is the only way to construct global-in-time solutions (except in a small-data setting)

Bourgain's invariant measure argument ('94):

Use a (formally) invariant measure in place of a conservation law

- use invariance of truncated Gibbs measures ρ_N and combine it with a PDE approximation argument
- mode of convergence of ρ_N should be sufficiently strong (such as "in total variation")

In the hyperbolic Φ^3_3-setting, the truncated Gibbs measures $\vec{\rho}_N$ converges *only weakly* to $\vec{\rho}$ with an uniform bound on the truncated densities *only in* $L^1(\mu)$

In short, Bourgain's invariant measure argument is *not* applicable to the hyperbolic Φ^3_3-model (but, of course, the philosophy still applies!!)

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Bringmann '20: introduced a new globalization argument in the context of the defocusing Hartree NLW on $\mathbb{T}^3$ with the Gibbs initial data (singular w.r.t GFF μ)

- Bourgain's invariant measure argument: "the probabilistic version of a deterministic global theory using a (sub-critical) conservation law"
- Bringmann studies the quantity $Q_{M,N}(t) = \vec{\rho}_M((u_N, \partial_t u_N)(t) \in A)$, where $(u_N, \partial_t u_N)$ is the solution to the truncated equation with a cutoff parameter N
  - $Q_{M,N}(t)$ is not conserved for $M \neq N$, but should be close to being constant in time when $M, N \gg 1$.
- Bringmann: "the probabilistic version of a deterministic global theory using almost conservation laws"
- relies on shifted measures (to which the truncated Gibbs measures are abs. conti.)

Oh-Okamoto-Tolomeo '21: a new, conceptually simple and straightforward approach, where we directly work with the (truncated) Gibbs measure

- uses extensively the **variational approach** (without referring to shifted measure)
- uses ideas from theory of optimal transport

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New globalization argument in Oh-Okamoto-Tolomeo '21 :

1. **Uniform bound on the truncated enhanced data sets.** Establish a uniform (in N) exponential integrability of the *truncated enhanced data set*

$$\Xi_N = \big(\uparrow_N, \, \vee_N, \, \curlyvee_N, \, \curlyvee^{\ominus}_N, \, \mathbb{A}_N, \, \mathfrak{I}^N_{\varolessthan, \ominus} \big),$$

   w.r.t. the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$ ($\mathbb{P}_2$ = measure on the noise). This is done by the *variational approach combined with space-time estimates* (such as the Strichartz estimates)

   We also show that Ξ_N converges to the limiting enhanced data set Ξ associated with the Gibbs measure $\vec{\rho}$ (almost surely with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$)

2. **Stability result.** By a simple contraction argument, where we use a norm with an *exponentially decaying weight in time*. (A small modification of the LWP argument)

3. **Uniform bound on the solutions** $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated hyperbolic Φ^3_3-system with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$

   The proof is based on the *invariance of the truncated Gibbs measure $\vec{\rho}_N$*

4. **Distributions of enhanced data sets.** $\mathcal{X}_T$ = space for (truncated) enhanced data sets

By using ideas from theory of optimal transport (the *Kantorovich duality*) and the *variational approach with space-time estimates* (as in Step 1), we prove that the pushforward measure $\nu_N = (\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ converges to $\nu = (\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$ in the *Wasserstein-1 distance*, as $N \to \infty$

More precisely: Fix $T > 0$. Then, there exists a sequence $\{\mathfrak{p}_N\}_{N\in\mathbb{N}}$ of probability measures on $\mathcal{X}_T \times \mathcal{X}_T$ with the first and second marginals ν and ν_N on $\mathcal{X}_T$, respectively:

$$\int_{\Xi^2\in\mathcal{X}_T} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu(\Xi^1) \qquad \text{and} \qquad \int_{\Xi^1\in\mathcal{X}_T} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu_N(\Xi^2)$$

such that

$$\int_{\mathcal{X}_T\times\mathcal{X}_T} \min(\|\Xi^1 - \Xi^2\|_{\mathcal{X}_T}, 1) d\mathfrak{p}_N(\Xi^1, \Xi^2) \longrightarrow 0, \quad \text{as } N \to \infty$$

These four steps yield

(i) a.s. global well-posedness and (ii) invariance of the Gibbs measure

Moral: What is important is the convergence of the truncated enhanced data sets (and not strong convergence of the truncated Gibbs measures $\vec{\rho}_N$)

Remarks and comments

1. Critical threshold value σ_{crit} for the construction of the Φ_3^3-measure?
   What happens at $\sigma = \sigma_{\text{crit}}$? (Focusing Φ_1^6-measure: Oh-Sosoe-Tolomeo '21)
2. Ergodicity? (Defocusing wave case: 1-d (Tolomeo '20), 2-d (Tolomeo '21+))
3. On $\mathbb{R}^3$? (1-d focusing case: Rider '02, Tolomeo-Weber '21)
4. Criticality of the Φ_4^4-measure "$\sim$" criticality for local well-posedness

   Criticality of the focusing Φ_1^6-measure or the Φ_3^3-measure
   "$\sim$" criticality for global well-posedness
5. focusing Hartree Φ_3^4-model
   - wave case: Oh-Okamoto-Tolomeo '20
   - Schrödinger case:
     - $\beta > 2$: Bourgain '97
     - $\beta = 2$ (weakly nonlinear regime): OOT '20 & Deng-Nahmod-Yue '21