Singular HJB with application to KPZ on the real line

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Motivation-(KPZ equation)

- Kardar-Parisi-Zhang (KPZ) equation on the real line $\mathbb{R}$:

$$\partial_t h = \Delta h + \text{“}(\partial_x h)^2\text{”} + \xi, \quad h(0) = h_0, \tag{1}$$

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$$\partial_t w = \Delta w + w\xi, \quad w(0) = \mathrm{e}^{h_0}. \tag{2}$$

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- Bertini and Giacomin (1997) called $\log w$ being the solutions to (1).
- It is not clear in what sense the Cole-Hopf solution solves the original KPZ equation.

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- Question: How to deduce suitable apriori estimates for general nonlinearity (Cole-Hopf not applicable)?
- Decompose $h = Y + h_1$ with

$$(\partial_t - \Delta)Y = \xi,$$

$$(\partial_t - \Delta)h_1 = (\partial_x h_1)^2 + 2\partial_x h_1 \partial_x Y + (\partial_x Y)^2.$$

Background-(Singular HJB equations)

- By a renormalization and decomposition procedure, one can reduce KPZ equation to the following HJB equation on $\mathbb{R}^+ \times \mathbb{R}^d$:

$$\partial_t u = \Delta u + H(\nabla u) + b \cdot \nabla u + f, \quad u(0) = u_0, \tag{3}$$

- Here $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, C^1-function, and for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\kappa \in (0, 1)$,

$$b \in L_T^\infty \mathbf{C}^{-\alpha}(\rho_\kappa), f \in L_T^\infty \mathbf{C}^{-\alpha}(\rho_\kappa),$$

where $\rho_\kappa(x) := \langle x \rangle^{-\kappa} := (1 + |x|^2)^{-\kappa/2}$.

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$$b \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa), f \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa),$$

  where $\rho_\kappa(x) := \langle x \rangle^{-\kappa} := (1 + |x|^2)^{-\kappa/2}$.
- (Ill-defined problem) $b \cdot \nabla u$ does not make sense since

$$\mathbf{C}^\alpha \times \mathbf{C}^\beta \ni (f, g) \to fg \in \mathbf{C}^{\alpha \wedge \beta} \text{ only if } \alpha + \beta > 0, \text{ where } \mathbf{C}^\alpha := B^\alpha_{\infty,\infty}.$$

- $\mathbf{C}^\alpha(\rho_\kappa) = \{f : f\rho_\kappa \in \mathbf{C}^\alpha\}$ (allow growth at infinity)

Decomposition

- Consider the HJB equation with distribution coefficients

$$\mathscr{L}u := (\partial_t - \Delta)\,u = H(\nabla u) + b \cdot \nabla u + f, \quad u(0) = u_0,$$

- Decompose HJB equation: linear equation with bad f and nonlinear equation without bad f

$$\mathscr{L}u_1 = b \cdot \nabla u_1 + f, \quad u_1(0) = u_0,$$

$$\mathscr{L}u_2 = b \cdot \nabla u_2 + H(\nabla(u_1 + u_2)), \quad u_2(0) = 0,$$

Linear equation

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$$\mathscr{L}_\lambda u := (\partial_t - \Delta + \lambda)u = b \cdot \nabla u + f, \quad u(0) = u_0. \tag{4}$$

- Suppose that for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and ρ_κ, $(b, f) \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)$.

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- Aim: Schauder estimate for (4) with solutions stay in the Besov space with sublinear growth weight.

Paracontrolled solution to linear PDE

- Paraproducts: if $f \in C^\alpha, g \in C^\beta$ for $\alpha > 0, \beta < 0$

$$fg = \underbrace{f \prec g}_{\text{bad term}} + \underbrace{f \circ g}_{\text{well defined only if } \alpha+\beta>0} + f \succ g,$$

  $f \prec\!\!\prec g$: modified paraproduct like $f \prec g$

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$$\mathscr{L}_\lambda u^\sharp = \nabla u \prec b - \nabla u \prec\!\!\prec b + \nabla u \succ b + b \circ \nabla u - [\mathscr{L}_\lambda, \nabla u \prec\!\!\prec] \mathscr{I}_\lambda b$$

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- $u \in C_T \mathbf{C}^{2-\alpha}(\rho_\delta), u^\sharp \in C_T \mathbf{C}^{3-2\alpha}(\rho_\delta)$
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- $u \in C_T\mathbf{C}^{2-\alpha}(\rho_\delta), u^\sharp \in C_T\mathbf{C}^{3-2\alpha}(\rho_\delta)$
- $b \circ \nabla \mathscr{I}_\lambda b, b \circ \nabla \mathscr{I}_\lambda f \in L^\infty_T\mathbf{C}^{1-2\alpha}(\rho_\kappa) \Rightarrow b \circ \nabla u \in L^\infty_T\mathbf{C}^{1-2\alpha}(\rho_\kappa)$ by commutator estimate and paracontrolled ansatz

Schauder estimate for linear PDE

$$(\partial_t - \Delta + \lambda)u = b \cdot \nabla u + f, \quad u(0) = u_0.$$

Theorem 1

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ *and* $\vartheta := \frac{9}{2-3\alpha}$ *and* $\delta := (2\vartheta + 2)\kappa \leqslant 1$. *For any* $T > 0$, (b, f) *as above,* $\exists!$ *paracontrolled solution* $(u, u^\sharp)$ *such that* $\|u\|_{C_T \mathbf{C}_T^{2-\alpha}(\rho_\delta)} + \|u^\sharp\|_{C_T \mathbf{C}_T^{3-2\alpha}(\rho_{2\delta})} \lesssim C(b, f)$.

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Idea of Proof

- Step 1: Schauder estimate for b, f in unweighted Besov space $\|u\|_{L_T^\infty \mathbf{C}^{2-\alpha}}$ depending polynomially on the coefficient (b, f) not exponentially as Gronwall

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- Step 2: Schauder estimate for b, f in weighted Besov space
  Trick: Localization+New characterization of Besov space

Step 1: Schauder estimate for b, f in unweighted Besov space

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- Schauder estimate: $\mathscr{I}_\lambda = \mathscr{L}_\lambda^{-1}$, $\theta \in (\alpha, 2]$

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- Choosing λ large enough $\Rightarrow$

$$\|u_\lambda\|_{L^\infty_T \mathbf{C}^{\theta-\alpha}} \lesssim \|f\|_{L^\infty_T \mathbf{C}^{-\alpha}}(\|b\|_{L^\infty_T \mathbf{C}^{-\alpha}} + 1) + \sup_\lambda \|b \circ \nabla \mathscr{I}_\lambda f\|_{L^\infty_T \mathbf{C}^{1-2\alpha}}.$$

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- General λ: $u = w + \bar{u}$

$$\begin{aligned} \partial_t w &= \Delta w - (\lambda' + \lambda)w + b \cdot \nabla w + f, \\ \partial_t \bar{u} &= \Delta \bar{u} - \lambda \bar{u} + b \cdot \nabla \bar{u} + \lambda' w, \end{aligned}$$

$\|u\|_{L_T^\infty \mathbf{C}^{2-\alpha}}$ depending polynomially on the coefficient (b, f) not exponentially as Gronwall

Step 2: Localization and characterization

- Let $\chi \in C_c^\infty(\mathbb{R}^d)$ with
$$\chi(x) = 1, \ \ |x| \leqslant 1/8, \quad \chi(x) = 0, \ \ |x| > 1/4,$$
For $r > 0$ and $z \in \mathbb{R}^d$, define
$$\chi_r^z(x) := \chi((x-z)/r), \ \ \phi_r^z(x) := \chi^z_{r(1+|z|)}(x).$$

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- Key point: $|\nabla \phi_r^z| \lesssim (1+|z|)^{-1} \Rightarrow$ gives extra weight $(1+|z|)^{-1}$

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- For each $z \in \mathbb{R}^d$, $u_z = u\phi_r^z$

$$\partial_t u_z = \Delta u_z + b_z \cdot \nabla u_z + F_z, \ \ u_z(0) = 0,$$

where $b_z := b\phi_{2r}^z$ and

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Lemma 2 (A characterization of weighted Hölder spaces)

Let $\alpha \geqslant 0$ and $r \in (0,1]$. For any $\delta, \kappa \in \mathbb{R}$, there is a constant $C = C(r, \alpha, d, \delta, \kappa) > 0$ such that for $\rho_\delta(z) := \langle z \rangle^{-\delta}$,

$$\|f\|_{\mathbf{C}^\alpha(\rho_\delta)} \asymp \sup_z \left(\rho(z)\|\chi_r^z f\|_{\mathbf{C}^\alpha}\right) \asymp_C \sup_z \left(\rho_\delta(z)\|\phi_r^z f\|_{\mathbf{C}^\alpha}\right).$$

Nonlinear Hamilton-Jacobi-Bellman equation

Nonlinear H-J-B equation: Zvonkin transform

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  $\Rightarrow \Phi(x) = x + \mathbf{u}$ is C^1-diffeomorphism

Zvonkin transform

- $v(t,x) = u_2(t, \Phi^{-1}(x)) \Rightarrow v(t, \Phi(t,x)) = u_2(t,x)$
- Recall

$$(\partial_t - \Delta)u_2 = b \cdot \nabla u_2 + H(\nabla(u_1 + u_2)).$$

$$(\partial_t - \Delta)\Phi = -\lambda \mathbf{u} + (b_> - \bar{b}_\leqslant) \cdot \nabla \Phi.$$

- By the chain rule,

$$\partial_t v \circ \Phi + \partial_t \Phi \cdot (\nabla v \circ \Phi) = \partial_t u_2, \ \ \nabla u_2 = \nabla \Phi \cdot (\nabla v \circ \Phi).$$

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$$(\partial_t v) \circ \Phi = \mathrm{tr}(a \cdot \nabla^2 v \circ \Phi) + H(u_1 + u_2, \nabla u_1 + \nabla u_2)$$
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$\Rightarrow \partial_t v = \mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + H(v, \nabla v)$, Good point: B, a: functions

General Hamilton-Jacobi-Bellman equation

- Consider the following non-divergence HJB equation in $\mathbb{R}^d$:

$$\partial_t v = \mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + H(v, \nabla v), \ \ v(0) = v_0,$$

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$$|H(t, x, v, Q)| \lesssim_{c_2} \langle x \rangle^\delta + |Q|^\zeta.$$

Theorem 3

For any initial value $v_0 \in \mathbf{C}^2(\rho_\delta)$, there are strong solution v for HJB equation such that for p large

$$\|v\|_{\mathbb{L}^\infty_T(\rho_\delta)} + \|\partial_t v\|_{\mathbb{L}^p_T(\rho_\eta)} + \|v\|_{\mathbb{H}^{2,p}_T(\rho_\eta)} \leqslant C.$$

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- Maximum principle for $w(t,x) := \mathrm{e}^{\lambda v(t,x)}$,

$$\partial_t w = \mathrm{tr}(a \cdot \nabla^2 w) + B \cdot \nabla w + \lambda w \underbrace{\Big(H(v, \nabla v) - \lambda \mathrm{tr}(a \cdot \nabla v \otimes \nabla v)\Big)}_{\leqslant \langle x \rangle^\delta \text{ for } \lambda \text{ large}}.$$

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- L^p-energy estimate for u+ $L^\infty(\rho_\delta)$ estimate
  Integration by parts and Hölder regularity of H

Singular HJB and main results

- $\mathscr{L}u = (\partial_t - \Delta)\, u = b \cdot \nabla u + H(u, \nabla u) + f, \quad u(0) = \varphi,$
  $b, f \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)$, $b \circ \nabla \mathscr{I}(b), b \circ \nabla \mathscr{I}(f) \in L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_\kappa)$, $\alpha \in (\frac{1}{2}, \frac{2}{3})$, κ small

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Theorem 4

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and κ be small enough so that $\delta := 2(\frac{9}{2-3\alpha} + 1)\kappa < 1$. Suppose that for some $c > 0$, and $\zeta \in (0, 2)$

$$|\partial_Q H(Q)| \leqslant c(1 + |Q|), d = 1; \quad |H(Q)| \leqslant c(|Q|^\zeta + 1), d \geqslant 2.$$

Then for (b, f) as above and $\varepsilon > 0$ initial value $u_0 \in \mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})$, there exists a unique paracontrolled solution $u \in L_T^\infty(\mathbf{C}^{2-\bar{\alpha}}(\rho_\eta))$ to HJB equation.

Applications

- Examples of H:

$$H(x,u) = g_1(x)|\nabla u|^2 + F(u)\nabla u + F_1(u) + g(x),$$

where $g_1 \in \mathbf{C}^\beta$, $\beta \in (0,1)$, $g \in L^\infty(\rho_\delta)$, and $F, F_1 \in \mathbf{C}^1$

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- Global well-posedness of KPZ equation/ Improved weight bound for KPZ equation
- Global well-posedness of the following two equations:

$$\mathscr{L}h=(\partial_x h)^2+g(h)+\xi,\quad h(0)=h_0,$$

$$\mathscr{L}h=G(x)(\partial_x h)^2+K(x)\partial_x h+\xi,\quad h(0)=h_0,$$

where g,G,K: bounded Lipschitz functions, ξ space-time white noise.

Thank you !