September 9, 2021
Conference on randomness and PDE

Transport of gaussian measures under Hamiltonian PDE's

N. Tzvetkov
CY Cergy Paris University

based on joint works with G. Genovese, T. Gunaratnam, R. Luca, Hiro Oh, F. Planchon, Ph. Sosoe, N. Visciglia, H. Weber.

The Sobolev spaces on the circle

• Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We denote by $H^s(\mathbb{T})$ the Sobolev spaces on the circle.

• If

$$u(x) = \sum_{n\in\mathbb{Z}} e^{inx}\,\widehat{u}(n),$$

where

$$\widehat{u}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-inx}\,u(x)dx \in \mathbb{C}$$

then

$$\|u\|_{H^s}^2 := \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s}|\widehat{u}(n)|^2,$$

where

$$\langle n\rangle := (1+n^2)^{\frac{1}{2}}.$$

• The norm H^s is induced from a natural scalar product which makes $H^s(\mathbb{T})$ a Hilbert space.

The gaussian measure μ_s

• We wish to define a gaussian measure of the form

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as a measure on a suitable functional space.

• Formally

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du = Z^{-1} \exp\Big(-\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\widehat{u}(n)|^2\Big) \prod_{n\in\mathbb{Z}} d\,\widehat{u}(n)$$

and the last expression makes think about the well defined object

$$\prod_{n\in\mathbb{Z}} Z_n^{-1} \exp\Big(-\langle n\rangle^{2s} |\widehat{u}(n)|^2\Big) d\,\widehat{u}(n),$$

where we formally wrote

$$Z^{-1} = \prod_{n\in\mathbb{Z}} Z_n{}^{-1} \quad (Z_n = \pi\langle n\rangle^{-2s}).$$

The gaussian measure μ_s (sequel)

- Therefore, we can define the measure μ_s

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as the image measure by the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}} e^{inx}\, \frac{g_n(\omega)}{\langle n\rangle^s}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are i.i.d. complex gaussian random variables with mean 0 and variances 1, on a probability space $(\Omega, \mathcal{F}, p)$.

- **Question :** μ_s is a measure on which space ?

The gaussian measure μ_s (sequel)

- We can write for $N < M$

$$\Big\| \sum_{N \leq |n| \leq M} e^{inx} \, \frac{g_n(\omega)}{\langle n \rangle^s} \Big\|^2_{L^2(\Omega; H^\sigma(\mathbb{T}))} \simeq \sum_{N \leq |n| \leq M} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < s - \frac{1}{2} \, .$$

- Therefore

$$\sum_{n \in \mathbb{Z}} e^{inx} \, \frac{g_n(\omega)}{\langle n \rangle^s} \in L^2(\Omega; H^\sigma(\mathbb{T})) \, .$$

The gaussian measure μ_s (sequel)

- We conclude that the map

$$\omega \longmapsto \sum_{n \in \mathbb{Z}} e^{inx} \, \frac{g_n(\omega)}{\langle n \rangle^s}$$

defines a probability measure on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$. In addition

$$\mu_s(H^{s-\frac{1}{2}}(\mathbb{T})) = 0 \, .$$

- In particular

$$\mu_s(H^s(\mathbb{T})) = 0 \, .$$

- In this constriction $H^s(\mathbb{T})$ is canonical but $H^\sigma(\mathbb{T})$ is not, it may be replaced for instance by $W^{\sigma,\infty}(\mathbb{T})$.

The Cameron-Martin theorem

• **Question :** How behaves μ_s under transformations ?

Theorem 1 (Cameron-Martin 1944)
Let $f \in H^\sigma(\mathbb{T})$ and let μ_f be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by

$$u \longmapsto f + u\,.$$

Then μ_f is absolutely continuous with respect to μ_s if and only if

$$f \in H^s(\mathbb{T}).$$

• Recalling that formally

$$d\mu_s(u) = Z^{-1}\, e^{-\|u\|^2_{H^s}}\, du$$

we may expect that

$$\frac{d\mu_f(u)}{d\mu_s(u)} = e^{-\|f\|^2_{H^s} - 2(u,f)_s}\ ,$$

where $(\cdot,\cdot)_s$ stands for the H^s scalar product.

Proof of the Cameron-Martin theorem for μ_s

• Let $f \in H^s(\mathbb{T})$. Since we expect that the Radon-Nykodim derivative is exp $\Big(-\|f\|^2_{H^s} - 2(u,f)_s\Big)$ the whole issue is to show that $(u,f)_s < \infty$, μ_s almost surely which is equivalent to

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^s \widehat{f}(n)\, \overline{g_n(\omega)} < \infty, \qquad \text{a.s.}$$

which directly results directly from the independence and $f \in H^s(\mathbb{T})$.
• Let now $f \notin H^s(\mathbb{T})$. Then there is $g \in H^s$ such that $(f,g)_s = \infty$. Consider the set

$$A = \{u \in H^\sigma \,:\, (g,u)_s < \infty\}.$$

We already checked that $\mu_s(A) = 1$ (replace f by g in the discussion of the first half of the slide). The image of A under our shift is the set B defined by

$$B = \{u + f, \quad u \in A\}.$$

Clearly $A \cap B = \emptyset$ and therefore $\mu_s(B) = 0$.

Thus we found a set of measure 1 which is sent by the shilt by f map to a set of measure 0. This completes the proof.

Invariance of μ_s under the free Schrödinger evolution

Proposition 2
Let $S(t)=e^{it\partial_x^2}$. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u\longmapsto S(t)(u)$. Then $\mu_s(t)=\mu_s$.

Proof. We have that

$$S(t)\Big(\sum_{n\in\mathbb{Z}} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}\Big)=\sum_{n\in\mathbb{Z}} e^{inx}\,\frac{e^{-itn^2}g_n(\omega)}{\langle n\rangle^s}$$

which has the same distribution as

$$\sum_{n\in\mathbb{Z}} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}$$

because $e^{-itn^2}g_n(\omega)$ has the same distribution as $g_n(\omega)$ (invariance of complex gaussians by rotations). This completes the proof.

A remark

• For a fixed sequence $(c_n)_{n\in\mathbb{Z}}$ the free Schrödinger evolution

$$\sum_{n\in\mathbb{Z}} c_n\, e^{inx}\, e^{-itn^2}$$

may have a complicated behaviour depending on the nature of the number t (leading to interesting number theory considerations) but the statistical behaviour under μ_s is the same for each time t.

Transport of μ_s under nonlinear transformations

Question : How behaves μ_s under the flow of the nonlinear Schrödinger equation (NLS) ? Let us start by the dispersionless model :

Theorem 3

Let $s \geq 1$ be an integer. Let $\rho_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves

$$i\partial_t u = |u|^4 u, \quad u|_{t=0} = u_0 \,. \tag{1}$$

Then for $t \neq 0$, the measure $\rho_s(t)$ is not absolutely continuous with respect to μ_s.

- The solution of (1) is given by

$$u(t,x) = u_0(x)\; e^{-it|u_0(x)|^4} \tag{2}$$

and the idea behind the proof is to show that a typical regularity property of the data resulting from the iterated logarithm law associated with μ_s is destroyed by the time oscillation in formula (2).

Transport of μ_s under nonlinear transformations (sequel)

But we also have :

Theorem 4

Let $s \geq 1$ be an integer. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves the nonlinear Schrödinger equation

$$(i\partial_t + \partial_x^2)u = |u|^4 u, \quad u|_{t=0} = u_0 \,. \tag{3}$$

Then $\mu_s(t)$ is absolutely continuous with respect to μ_s. In other words, μ_s is quasi-invariant under the flow of (3).

- We have similar results for the fractional NLS in $1d$, for the nonlinear wave equations in dimensions ≤ 3, for the gKdV equation and for BBM type models.
- Depending on the equation, we have more or less informations on the resulting Radon-Nykodim derivatives.
- I am very interested in the extension to the $2d$ NLS which seems to require some new ingredients. Even the $3d$ NLS does not seem completely out of reach ...

A corollary (L^1 stability for the corresponding Liouville equation)

Theorem 5

Let $s \geq 1$ be an integer. Let $f_1, f_2 \in L^1(d\mu_s)$ and let $\Phi(t)$ be the flow of

$$(i\partial_t + \partial_x^2)u = |u|^4 u, \quad u|_{t=0} = u_0 \,,$$

defined μ_s a.s. Then for every $t \in \mathbb{R}$, the transports of the measures

$$f_1(u)d\mu_s(u), \quad f_2(u)d\mu_s(u)$$

by $\Phi(t)$ are given by

$$F_1(t,u)d\mu_s(u), \quad F_2(t,u)d\mu_s(u)$$

respectively, for suitable $F_1(t,\cdot), F_2(t,\cdot) \in L^1(d\mu_s)$. Moreover

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_s)} = \|f_1 - f_2\|_{L^1(d\mu_s)} \,.$$

• Local in time bounds for other distances are obtained in a recent work by work by Forlano-Seong. There are many further things to be understood.

Remarks

• The above results are restricted to relatively regular solutions of the equation (cf. the assumption $s \geq 1$) because the question of quasi-invariance seems *strictly more complicated* than the question of proving the existence of the dynamics (this seems to be an infinite dimensional phenomenon).

• For exemple, in the context of the impressive recent results by Deng-Nahmod-Yue for NLS with low regularity gaussian data, the question of the propagation of the gaussianity by the flow of the equation seems completely open.

• A similar remark applies to the earlier probabilistic well-posedness results by Nicolas Burq and myself on the nonlinear wave equation and by Colliander-Oh on the 1d NLS.

• I however expect that the methods and the ideas developed in the work on probabilistic well-posedness may become useful in quasi-invariance questions. Ideally, one day we will may be succeed to have a quasi-invariance result for a deterministically ill-posed posed problem.

Methods

• Roughly speaking, presently, we have two different methods to prove this kind of quasi-invariance results :

• **Method 1 :** Using the *time oscillations* (dispersive estimates).

• **Method 2 :** Using the *random oscillations* (concentration of measure estimates).

• In both methods, we do not study directly the evolution of the gaussian measure μ_s but the evolution of ρ_s defined by

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where $R_s(u)$ is a suitable correction and where χ is a continuous function with a compact support and where $H(u)$ is the Hamiltonian of the equation under consideration (conserved by the flow). We formally have

$$e^{-R_s(u)} d\mu_s(u) = Z^{-1} e^{-R_s(u)} e^{-\|u\|^2_{H^s}} du = Z^{-1} e^{-E_s(u)} du\,,$$

where

$$E_s(u) = \|u\|^2_{H^s} + R_s(u)\,.$$

Methods (sequel)

• The correction $R_s(u)$ in the energy functional

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

is of fundamental importance and there are different intuitions behind its construction : normal form reductions, traces of complete integrability, modulated energies, ...

• Interestingly, in some cases the construction of $R_s(u)$ requires renormalisation arguments (as we saw in the talk by F. Otto yesterday).

• However, an important feature is that we *do not renormalise the equation which stays always the same*. Instead, we consider renormalised functionals associated with the equation with data distributed according to a gaussian field.

On method 1

- Let $\Phi(t)$ be the flow of the PDE under consideration.
- Formally the transported measure is given by

$$Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, du = \\ Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, e^{E_s(u)}e^{-E_s(u)}du$$

which can be interpreted as the (relatively) well defined object

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}\,\chi(H(u))e^{-R_s(u)}d\mu_s(u)\,.$$

- Therefore we hope that the Radon-Nykodim derivative of the transport of ρ_s is given by

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}$$

- **Problem :** In $E_s(\Phi(t)(u))-E_s(u)$ both terms are strongly diverging on the support of μ_s but the hope is to find some cancellations thanks to PDE smoothing estimates.

On method 1 (sequel)

• More precisely, one can write

$$E_s(\Phi(t)(u)) - E_s(u) = \int_0^t \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau} d\tau.$$

Set

$$G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau}.$$

We will be done, if we can prove that

$$\Big| \int_0^t G_s(\tau) d\tau \Big| \leq C_{H(u)} \|u\|^{\theta}_{H^{s-\frac{1}{2}-}},$$

for a suitable choice of $R_s(u)$ and for a suitable number θ.

• If E_s is a conserved quantity (Gibbs measures) then $G_s = 0$ and one expects an invariant measure. However, this may not be true at the level of the approximated finite dimensional models and a serious difficulty may appear (cf. works by Nahmod-Oh-Rey Bellet-Staffilani, Tz.-Visciglia, Genovese-Luca-Valeri, ...).

On method 1 (sequel)

• If $\theta < 2$ the Randon-Nykodim density is indeed given by

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}$$

in the sense that it is the natural limit of the corresponding (perfectly well defined) finite dimensional densities.

• If $\theta \geq 2$, we can define the Radon-Nykodim density of the transport of

$$\exp\left(-\|u\|^m_{H^{s-\frac{1}{2}-}}\right)\chi(H(u))\,e^{-R_s(u)}d\mu_s(u),$$

where $m \gg 1$ (depending on θ).

• **Remark.** It would be interesting to replace

$$\left|\int_0^t G_s(\tau)d\tau\right| \leq C_{H(u)}\|u\|^\theta_{H^{s-\frac{1}{2}-}},$$

with more subtle estimates.

On method 2

- Let $A \subset H^\sigma(\mathbb{T})$ be a measurable set.
- Recall that

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where χ is a continuous function with a compact support and $H(u)$ is the Hamiltonian of the equation under consideration.

- Then

$$\frac{d}{dt}\rho_s(\Phi(t)(A))\Big|_{t=\bar{t}} = \frac{d}{dt}\rho_s(\Phi(t)(\Phi(\bar{t})(A)))\Big|_{t=0}$$

which is **formally** equal to

$$\int_{\Phi(\bar{t})(A)} \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0}\, d\rho_s(u)$$
$$\leq \left\| \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0} \right\|_{L^p(\rho_s)} \Big(\rho_s(\Phi(\bar{t})(A))\Big)^{1-\frac{1}{p}}$$

On method 2 (sequel)

• We would be done if we show that

$$\left\| \frac{d}{dt} E_s(\Phi(t)(A)) \Big|_{t=0} \right\|_{L^p(\rho_s)} \leq Cp, \quad p \gg 1\,. \tag{4}$$

In the proof of the last inequality we only exploit the random oscillations of the initial data.

• Important observation : if we are only interested in the qualitative statement of quasi-invariance then in (4) we can suppose that A included in a bounded set of a Banach space $\mathcal{H}$ which is of full measure such that the PDE under consideration is globally well posed in $\mathcal{H}$ (existence, uniqueness and persistence of regularity).

• Let us **formally** show how we use (4) (similarly to the uniqueness for $2d$ Euler) to get the quasi-invariance. Set

$$x(t) = \rho_s(\Phi(t)(A))\,.$$

Thanks to (4) we have

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}$$

On method 2 (sequel)

Therefore

$$\frac{d}{dt}\Big((x(t))^{\frac{1}{p}}\Big) \leq C\,.$$

• An integration yields

$$(x(t))^{\frac{1}{p}} - (x(0))^{\frac{1}{p}} \leq Ct$$

Therefore, if $x(0) = 0$ then

$$x(t) \leq (Ct)^p$$

which goes to zero as $p \to \infty$, provided $Ct < 1$.

• Since the constant C is uniform we can iterate the last argument and achieve any time.

• The above argument may become rigorous if we use some approximation arguments resulting from the Cauchy problem theory of the equation under consideration.

A final remark

- Method 2 performs better for equations with weak dispersion.
- It would be interesting to find a way to combine Method 1 and Method 2 ...

Thank you for your attention !