Phase transitions of the focusing Φ_1^p measures

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Based on joint works with T. Oh and P. Sosoe & H. Weber.

Mass (sub-)critical focusing NLS

Consider the *focusing* Schrödinger equation

$$\text{(NLS)} \qquad \begin{cases} iu_t - \partial_x^2 u - \sigma^2 |u|^{p-2} u = 0 \\ u(0) = u_0, \end{cases}$$

posed on $\mathbb{T}$.

We have the following conserved quantities:

$$M(u) = \int |u|^2 dx,$$

$$E(u) = \frac{1}{2} \int |\partial_x u|^2 - \frac{\sigma^2}{p} \int |u|^p dx.$$

$M(u)$ is coercive, while $E(u)$ is not coercive.

NLS: long time behaviour

$$M(u) = \int |u|^2 dx, \qquad E(u) = \frac{1}{2}\int |\partial_x u|^2 - \frac{\sigma^2}{p}\int |u|^p dx.$$

Long time behaviour:

- If $p < 6$, the equation is *mass subcritical*, and solutions exist for infinite time.

$$\int |u|^p \lesssim \overbrace{M(u)^{\frac{p+2}{4}}}^{\text{conserved}} \left(\int |\partial_x u|^2\right)^{\overbrace{\frac{p-2}{4}}^{<1}} \rightsquigarrow E(u)\mathbb{1}_{M\leqslant K} \text{ coercive.}$$

- If $p = 6$, the equation is *mass-critical*. There are two behaviours.
  - If $M(u) < \sigma^{-1}K_0$, solutions exist for infinite time.
  - If $M(u) \geqslant \sigma^{-1}K_0$, there are solutions that blow up in finite time.

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  - If $M(u) < \sigma^{-1} K_0$, solutions exist for infinite time.
  - If $M(u) \geqslant \sigma^{-1} K_0$, there are solutions that blow up in finite time.

$$\frac{\sigma^2}{6}\int_{\mathbb{R}} |u|^6 \leqslant \overbrace{C\sigma^2 M(u)^2}^{<\frac{1}{2}?} \left(\int_{\mathbb{R}} |\partial_x u|^2\right)$$

$$\Rightarrow K_0 \text{ s.t. } C^2 {K_0}^2 = \frac{1}{2}.$$

Focusing Φ_1^p measure

We want to build the gran-canonical ensemble for the focusing NLS,

$$\rho_{\sigma,p} = \text{``}\underbrace{\frac{1}{Z}}_{\text{normalisation constant}} \exp\Big(\overbrace{\frac{\sigma^2}{p}\int |u|^p}^{\text{density}} \overbrace{-\frac{1}{2}\int |\partial_x u|^2 - \frac{1}{2}\int |u|^2}^{\text{Gaussian}}\Big) du\text{''}.$$

Focusing sign $\Rightarrow$ not definable. In dimension 1:

$$\int_{\mathbb{R}} \exp\Big(+\frac{\sigma^2}{p}\int |g|^p - C\int |g|^2\Big) dg = \infty.$$

Lebowitz-Rose-Speer '88:

$M(u)$ is conserved $\rightsquigarrow$ we can introduce a mass cutoff $\mathbb{1}_{\{M(u)\leqslant K\}}$.

Focusing Φ_1^p measure

We want to build the "generalised grand-canonical Gibbs measure"

$$\rho_{\sigma,p,K} = \text{“}\frac{1}{Z}\exp\Big(\overbrace{\frac{\sigma^2}{p}\int|u|^p}^{\text{density}}\ \overbrace{-\frac{1}{2}\int|\partial_x u|^2-\frac{1}{2}\int|u|^2}^{\text{Gaussian}}\Big)\overbrace{\mathbb{1}_{\{M(u)\leqslant K\}}}^{\text{cutoff}}du\text{”}.$$

Rigorously, we fix the Gaussian measure μ with inverse covariance $1-\partial_x^2$, and define

$$\rho_{\sigma,p,K} = \frac{1}{Z(\sigma,p,K)}\exp\Big(\frac{\sigma^2}{p}\int|u|^p\Big)\mathbb{1}_{\{M(u)\leqslant K\}}d\mu(u).$$

$$\rho_{\sigma,p,K} \text{ definable } \longleftrightarrow Z(\sigma,p,K)<\infty.$$

Invariant measures $\leftrightarrow$ global well posedness, so we expect $Z<\infty \Leftrightarrow K<K_0$.

Existence and non existence of the Φ_1^p measure

Theorem: Lebowitz-Rose-Speer '88, Bourgain '94, Oh-Sosoe-T. '21, T.-Weber '21+

Let $2 < p \leqslant 6$, and let

$$Z(\sigma, p, K) := \int \exp\Big(\frac{\sigma^2}{p}\int |u|^p\Big)\mathbb{1}_{\{M(u)\leqslant K\}}d\mu(u).$$

Then, $Z(\sigma, p, K) < \infty$ in the following cases:

1. $K \in \mathbb{R}$, $p < 6$ (LRS '88*, B '94),
2. $p = 6$, $K < \sigma^{-1}K_0$ (LRS '88*, B '94 for $K \ll 1$, OST '21, TW '21+),
3. $p = 6$, $K = \sigma^{-1}K_0$ (OST '21),

and $Z(\sigma, p, K) = \infty$ in the following cases:

4. $p = 6$, $K > \sigma^{-1}K_0$ (LRS '88),
5. $p > 6$ (LRS '88).

Corollary: $Z(\sigma, 6, K)$ is *not* analytic in the parameters.

Carlen - Frölich - Lebowitz '16: $Z(\sigma, p, K)$ is analytic in σ, K for $p < 6$.

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3. $p = 6$, $K = \sigma^{-1}K_0$ (OST '21), $\longleftarrow$ measure doesn't see blowup data!

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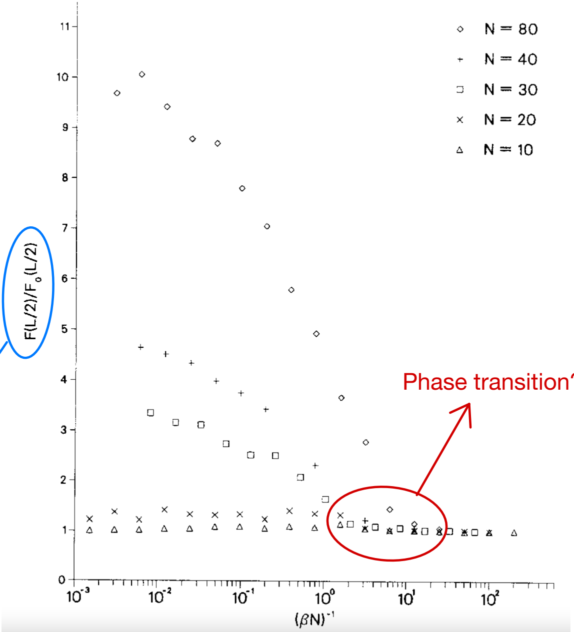
Other phase transitions? Big torus limit

LRS '88:



Measure of non-gaussianity

$\beta \sim \sigma^{-2}$

$N \sim K \sigma$

Big torus limit

For simplicity, let $p = 4$.

Theorem: Rider '02

Let $\mathbb{T}_L := \mathbb{R}/L\mathbb{Z}$ be the torus of size L, and consider the measure

$$\rho_{K,\sigma,L} = \exp\left(\frac{\sigma^2}{4}\int_{L\mathbb{T}}|u|^4\right)\mathbb{1}_{\{M(u)\leqslant KL\}}d\mu_L(u).$$

Then, as $L \to \infty$, $\rho_{K,\sigma,L} \rightharpoonup \delta_0$. More precisely, for any test function $f \in \mathcal{D}(\mathbb{R})$,

$$\lim_{L\to\infty}\mathbb{E}_{\mu_L}\left|\int_{L\mathbb{T}} f(x)u(x)dx\right| = 0.$$

$\Rightarrow$ the limit is trivial, and it has no phase transition.

No phase transition?

Rider's result proves that

$$\rho_{K,\sigma,L} = \exp\left(\frac{\sigma^2}{4}\int_{L\mathbb{T}} |u|^4\right)\mathbb{1}_{\{M(u)\leqslant KL\}}d\mu_L(u).$$

has no phase transition when $L \to \infty$.

LRS numerics "see" a phase transition depending on βN, where

$$\beta \sim \sigma^{-2}, \quad N \sim \frac{KL}{\beta}.$$

In Rider's result,

$$\beta N \sim \frac{KL\sigma^4}{\sigma^2} = KL\sigma^2 \sim L \gg 1.$$

Therefore, we should have $\sigma^2 \sim L^{-\gamma(p)}$.

Phase transition on large torus/high temperature regime

Theorem: T. - Weber '21+

Let $2 < p < 6$, $K > 2^{-\frac{3}{2}}$, and consider the measure

$$\rho_{K,\sigma,L,\gamma} = \exp\left(\frac{\sigma^2}{pL^\gamma}\int_{L\mathbb{T}} |u|^p\right)\mathbb{1}_{\{M(u)\leqslant KL\}}d\mu_L(u).$$

1. If $\gamma < \gamma_0(p)$, then
$$\lim_{L\to\infty} \rho_{K,\sigma,L,\gamma} = \delta_0.$$
2. If $\gamma > \gamma_0(p)$, then
$$\lim_{L\to\infty} \rho_{K,\sigma,L,\gamma} = \mu_\infty,$$
where μ_∞ denotes the Ornstein-Uhlenbeck process on $\mathbb{R}$.
3. If $\gamma = \gamma_0(p)$, we are in the LRS numerics case.
   - If $\sigma \ll 1 \leftrightarrow (\beta N)^{-1} \gg 1$,
   $$\lim_{L\to\infty} \rho_{K,\sigma,L,\gamma} = \mu_\infty.$$
   - If $\sigma \gg 1 \leftrightarrow (\beta N)^{-1} \ll 1$,

??? $\lim \neq \mu_\infty$.

Higher dimension

Theorem: Brydges - Slade '96, Oh - Seong - T. '20

The focusing Φ_2^4 measure

$$\exp\left(\frac{1}{2k}\int_{\mathbb{T}^2} : u^4 : dx\right)\mathbb{1}_{\{:M(u):\leqslant K\}}d\mu(u)$$

is not normalisable for $k > 1$, regardless of the value of K.

Theorem: Oh - Okamoto - T. '21

The Φ_3^3 measure

$$\exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^2} : u^3 : dx\right)\mathbb{1}_{\{:M(u):\leqslant K\}}d\mu(u)$$

is normalisable for $\sigma \ll 1$, and not normalisable for $\sigma \gg 1$.

Below the threshold: ingredient I

Suppose we have

$$\frac{\sigma^2}{p}\int_{\mathbb{T}}|u|^p \leqslant \big(1+\varepsilon\big)^{-1}\cdot\frac{1}{2}\int_{\mathbb{T}}|\partial_x u|^2 + C_E.$$

for $u \in E$. Then, by the Buoé-Dupuis formula, following Barashkov-Gubinelli:

$$\begin{aligned}
&-\log\int \exp\Big(\frac{\sigma^2}{p}\int|u|^p\Big)\mathbb{1}_E d\mu \\
&\geqslant \inf_{V\in H^1}\mathbb{E}_\mu\Big[-\mathbb{1}_E(u+V)\Big(\frac{\sigma^2}{p}\int|u+V|^p+\frac{1}{2}\int|\partial_x V|^2\Big)\Big] \\
&\geqslant \inf_{V\in H^1}\mathbb{E}_\mu\Big[-\frac{\sigma^2}{p}\int|u+V|^p+(1+\varepsilon)\frac{\sigma^2}{p}\int|V|^p+\text{error}\Big] \\
&> -\infty.
\end{aligned}$$

Threshold case for $p = 6$

Fix $\sigma = 1$. If $M(u) \leqslant K_0$,

$$\frac{1}{6}\int_{\mathbb{R}} |u|^6 \leqslant \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2.$$

Main obstacle: u such that we have "almost" $=$.

Let $Q \in S(\mathbb{R})$ be the minimal nontrivial solution of

$$\Delta Q - 2Q + Q^5 = 0.$$

Then $M(Q) = K_0$, and

$$\frac{1}{6}\int_{\mathbb{R}} |Q|^6 = \frac{1}{2}\int_{\mathbb{R}} |\partial_x Q|^2.$$

Moreover, if W satisfies $M(W) \leqslant K_0$, $\int |W|^6 = 3\int |\partial_x W|^2$, then

$$W(x) = e^{i\theta}\delta^{-\frac{1}{2}} Q(\delta^{-1}(x - x_0)) =: e^{i\theta} Q_{\delta, x_0}.$$

Stability of the optimisers: ingredient II

If $u \in L^2$ satisfies (in a suitable sense)

$$\int_{\mathbb{T}} |u|^6 > \left(\frac{1}{2} - \gamma\right) \int_{\mathbb{T}} |\partial_x u|^2 + C(\|u\|_{L^2}),$$

then there exist $\theta \in \mathbb{R}$, $x_0 \in \mathbb{T}$, $\delta \ll 1$ such that

$$\|u - e^{i\theta} Q_{\delta,x_0}\|_{L^2(\mathbb{T})} \leqslant \varepsilon(\gamma).$$

By the "under the threshold" proof,

$$Z < \infty \iff \mathbb{E}_\mu\Big[\exp\Big(\frac{1}{6}\int_{\mathbb{T}} |u|^6\Big)\mathbb{1}_{\{M(u)\leqslant K_0, \|u - e^{i\theta}Q_{\delta,x_0}\|_{L^2(\mathbb{T})}\leqslant \varepsilon\}}\Big] < \infty.$$

Threshold case for $p = 6$

We want to estimate

$$\text{“}\int \exp\left(\frac{1}{6}\int_{\mathbb{T}} |u|^6 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2\right) du\text{”}$$

on a neighbourhood of $e^{i\theta}Q_{\delta,x_0}$ in $M(u) \leqslant K_0$. Changing variables, we want to estimate

$$\begin{aligned}
&\exp\left(\frac{1}{6}\int_{\mathbb{T}} |Q_\delta + u|^6 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x(Q_\delta + u)|^2\right) \\
&= \exp\left(\frac{1}{6}\int_{\mathbb{T}} Q_\delta^6 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x Q_\delta|^2\right) \\
&\quad \times \exp\left(\int_{\mathbb{T}} (Q_\delta^5 - Q_\delta'')u\right) \\
&\quad \times \exp\left(\frac{5}{2}\int_{\mathbb{T}} Q_\delta^4 u^2 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2\right) \times \text{ l.o.t.}
\end{aligned}$$

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&\exp\Big(\frac{1}{6}\int_{\mathbb{T}} |Q_\delta + u|^6 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x(Q_\delta + u)|^2\Big) \\
&= \exp\Big(\frac{1}{6}\int_{\mathbb{T}} Q_\delta^6 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x Q_\delta|^2\Big) && \approx 1 \\
&\times \exp\Big(\int_{\mathbb{T}} (Q_\delta^5 - Q_\delta'')u\Big) && \approx -\delta^{-2}\int_{\mathbb{T}} |u|^2 \\
&\times \exp\Big(\frac{5}{2}\int_{\mathbb{T}} Q_\delta^4 u^2 - \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2\Big) \times \text{ l.o.t.}
\end{aligned}$$

← use the eqn. for Q.

Spectral analysis: ingredient III?

Reduced to estimate

$$\mathbb{E}_\mu\Big[\exp\Big(-\langle u, Au\rangle_{L^2} + \text{ error}\Big)\Big],$$

where

$$A = \delta^{-2} - \frac{5}{2}Q_\delta^4.$$

We need

$$\text{tr}\Big|\log\Big(\overbrace{1 + 2\partial_x^{-1}A\partial_x^{-1}}^{B}\Big)\Big| < +\infty.$$

Problem: Symmetries.

$$B\partial_\delta Q_\delta = 0, \quad B\partial_{x_0}Q_\delta = 0, \quad B\partial_\theta Q_\delta = 0.$$

$\Rightarrow \text{tr} = +\infty$.

Orthogonal decomposition: ingredient III

Lemma

There exists $\varepsilon_0 > 0$ such that, given $\theta \in [0, 2\pi], \delta \leqslant 1, x_0 \in \mathbb{T}$, if $\|u - e^{i\theta} Q_{\delta,x_0}\|_{L^2} < \varepsilon_0$, there exists a (unique) representation

$$u = e^{i\theta} Q_{\delta,x_0} + v,$$

where

$$v \perp_{H^1} V_{\theta,\delta,x_0} = \operatorname{span}(\partial_\theta e^{i\theta} Q_{\delta,x_0}, \partial_\delta e^{i\theta} Q_{\delta,x_0}, \partial_{x_0} e^{i\theta} Q_{\delta,x_0}),$$
$$\|v\|_{L^2} \lesssim \|u - e^{i\theta} Q_{\delta,x_0}\|_{L^2}.$$

This allows us to disintegrate the measure μ:

$$\mathbb{E}_\mu[F(u)] = \int_0^{2\pi} \int_0^1 \int_{\mathbb{T}} \mathbb{E}_{\mu^\perp_{\theta,\delta,x_0}} \left[F(e^{i\theta} Q_{\delta,x_0} + v)\right] d\sigma(\theta, \delta, x_0),$$

where $\mu^\perp_{\theta,\delta,x_0}$ is the measure on $V^\perp_{\theta,\delta,x_0}$ with covariance $(1 - \partial_x^2)^{-1}$.

Spectral analysis: Ingredient IV

We need to estimate

$$\mathbb{E}_{\mu_\delta^\perp}\Big[\exp\Big(-\langle u, Au\rangle_{L^2} + \text{ error}\Big)\Big],$$

where

$$A = \delta^{-2} - \frac{5}{2}Q_\delta^4.$$

On $V_\delta^\perp$, $B := 1 + 2\partial_x^{-1}A\partial_x^{-1}$ is strictly positive. Moreover,

$$A \gtrsim T := \delta^{-2}\begin{cases} -1 & \text{on } [-\delta/100, \delta/100], \\ 1 & \text{on } [-\delta/100, \delta/100]^c, \end{cases}$$

whose spectrum can be found explicitly. We obtain

$$\mathbb{E}_{\mu_\delta^\perp}\Big[\exp\Big(-\langle u, Au\rangle_{L^2} + \text{ error}\Big)\Big] \lesssim \exp(-c\delta^{-1}).$$

Optimal threshold recipe

Ingredients:

1. Non-optimal case:

$$\frac{\sigma^2}{p}\int_{\mathbb{T}}|u|^p \leqslant \Big(1+\gamma\Big)^{-1}\cdot\frac{1}{2}\int_{\mathbb{T}}|\partial_x u|^2 + C$$

2. L^2 - stability of the optimisers:

$$\|u - e^{i\theta}Q_{\delta,x_0}\|_{L^2} > \varepsilon \Rightarrow \text{ non-optimal case.}$$

3. H^1-Orthogonal decomposition of an L^2-neighbourhood of $\{e^{i\theta}Q_{\delta,x_0}\}_{\theta,\delta,x_0}$ + corresponding disintegration of μ in $\mu^{\perp}_{\theta,\delta,x_0}$.
4. Spectral analysis of

$$1+2\partial_x^{-1}\Big(\delta^{-2}-\frac{5}{2}Q_\delta^4\Big)\partial_x^{-1} \qquad \text{on } V_\delta^{\perp}.$$

Big torus limit

Define

$$A(\sigma, K) := \max\left\{\frac{\sigma^2}{4}\int_{\mathbb{R}}|u|^4 - \frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 : M(u) \leqslant K\right\},$$

and let $Q = Q_{\sigma,K}$ be an optimiser for $A(\sigma, K)$.

Almost soliton case: $\gamma < 1$.

$$\rho_L := \frac{1}{Z_L}\exp\left(\frac{\sigma^2}{4L^\gamma}\int_{L\mathbb{T}}|u|^4\right)\mathbb{1}_{\{M(u)\leqslant KN\}}\mu_L \rightharpoonup \delta_0.$$

1. $$Z_L \sim \exp\left(A(\sigma, K)L^{3-2\gamma}\right)$$
2. $$Z_L(\eta) := \mathbb{E}_{\mu_L}\left[\exp\left(\frac{\sigma^2}{4L^\gamma}\int_{L\mathbb{T}}|u|^4\right)\mathbb{1}_{\{M(u)\leqslant KN\}}\mathbb{1}_{\{\|L^{-\gamma/2}u(L^{1-\gamma}\cdot)-Q(\cdot-x_0)\|>\eta\}}\right]$$
$$\lesssim \exp\left((A(\sigma, K) - \varepsilon(\eta))L^{3-2\gamma}\right)$$

Therefore,

$$\rho_L\mathbb{1}_{\{\text{not soliton-like}\}} \lesssim \frac{Z_L(\eta)}{Z_L} \to 0 \text{ as } L \to \infty.$$

Big torus limit: $\gamma < 1$, main ingredients

1. Cameron-Martin theorem:

$$u = L^{\frac{\gamma}{2}} Q(L^{-(1-\gamma)} \cdot) + v$$

2. Stability of optimisers:

Let

$$A_\eta(\sigma, K) := \sup \left\{ \frac{\sigma^2}{4} \int_{\mathbb{R}} |u|^4 - \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 : M(u) \leqslant K, \|u - Q(\cdot - x_0)\| \leqslant \eta \right\}.$$

Then $A_\eta(\sigma, K) < A(\sigma, K)$ for every $\eta > 0$.

In particular, if u_n is a sequence of almost optimisers for $A(\sigma, K)$, we must have $u_n(\cdot - x_n) \to Q$ for some sequence x_n.

Big torus limit: $\gamma = 1, \sigma \ll 1$

Fix $\gamma = 1$, $\sigma \ll 1$. We want to show that

$$\rho_L \rightharpoonup \mu_\infty.$$

1. Using translation invariance, for $\theta < 1$,

$$\rho_L \approx \frac{1}{Z_L} \exp\left(\frac{\sigma^2}{4L}\int_{[-L^\theta, L^\theta]^c} |u|^4\right) \mathbb{1}_{\{M(u)\leqslant KN\}} \mu_L$$

2. Since $\sigma \ll 1$,

$$\begin{aligned}
&\frac{1}{Z_L} \exp\left(\frac{\sigma^2}{4L}\int_{[-L^\theta, L^\theta]^c} |u|^4\right) \mathbb{1}_{\{M(u)\leqslant KN\}} \mu_L \\
&\approx \frac{1}{Z_L} \exp\left(\frac{\sigma^2}{4L}\int_{[-L^\theta, L^\theta]^c} |u|^4\right) \mathbb{1}_{\{M(u|_{[-L^\theta, L^\theta]^c})\leqslant KN\}} \mu_L =: \rho_L^\theta
\end{aligned}$$

Big torus limit: $\gamma = 1, \sigma \ll 1$

3. Fix $K \subset \mathbb{R}$ compact, and let F be a "nice" functional.

$$\int \exp\Big(F(u|_K)\Big) d\rho_L^\theta \approx \mathbb{E}_{\mu_L}\Big[\exp\Big(F(u|_K)\Big)\Big] \times \int d\rho_L^\theta.$$

Based on short range interaction of μ_L: if $d(H,K) \gg 1$, then

$$\mathbb{E}_{\mu_L}\Big[\exp\Big(F(u|_K) + G(u|_H)\Big)\Big] \approx \mathbb{E}_{\mu_L}\Big[\exp\Big(F(u|_K)\Big)\Big]\mathbb{E}_{\mu_L}\Big[\exp\Big(G(u|_H)\Big)\Big].$$

Boué - Dupuis formula: if S closed,

$$\begin{aligned}
&-\log \mathbb{E}_{\mu_L}\Big[\exp\Big(-H(u|_S)\Big)\Big] \\
&= \inf_{V \text{ adapted}} \mathbb{E}\Big[H(u + V(1)) + \frac{1}{2}\int_0^1 \int_{L\mathbb{T}} \dot{V}(1)(1-\partial_x^2)\dot{V}(1)\Big].
\end{aligned}$$

By first variation consideration, the optimiser V must satisfy $V - V'' = 0$ on $S^c \rightsquigarrow$ exponential decay outside of S.

Big torus limit: $\gamma = 1, \sigma \gg 1$?

Thank you for your attention!