Path functions and homogenisation

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(based on joint works with P. Friz, A. Korepanov & I. Melbourne)

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Overview

1. Differential equations with jumps
2. Generalisation of Skorokhod M_1 topology
3. Applications to Marcus SDEs
4. Homogenisation of superdiffusive fast-slow systems

Differential equations with jumps

Jumps - interpretation

Let $D \equiv D([0,T],\mathbb{R}^d)$ denote space of càdlàg paths, i.e. functions $X\colon [0,T] \to \mathbb{R}^d$ such that

$$X_{t-} := \lim_{s\uparrow t} X_s \;\text{ exists,}\quad \text{and} \quad \lim_{s\downarrow t} X_s = X_t \;.$$

Question: how to interpret the jumps of $X \in D$?

- True discontinuity vs.
- Unobserved continuous movement.

Different interpretations lead to completely different solutions to ODEs

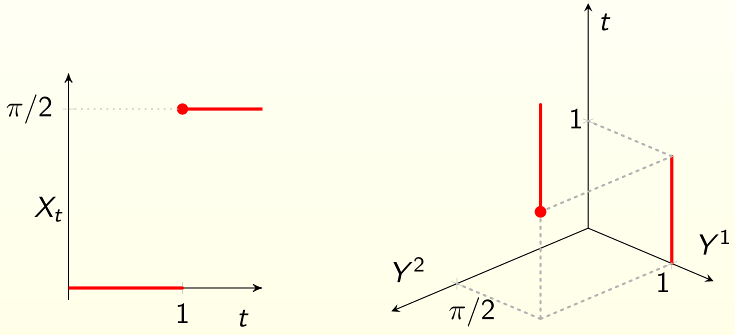
$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}X_t$$

- Forward (Itô) equation $\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}X_t$ (natural e.g. in finance) vs.
- Geometric (Marcus) equation $\mathrm{d}Y_t = b(Y_t)\diamond\mathrm{d}X_t$ (natural e.g. in physics and geometry).

Consider $X_t = \frac{\pi}{2}\mathbf{1}_{t\geq 1}$ and the ODE

$$\begin{pmatrix} \mathrm{d}Y^1 \\ \mathrm{d}Y^2 \end{pmatrix} = \begin{pmatrix} -Y^2 \\ Y^1 \end{pmatrix} \mathrm{d}X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \mathrm{d}X \,, \quad \begin{pmatrix} Y_0^1 \\ Y_0^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,.$$

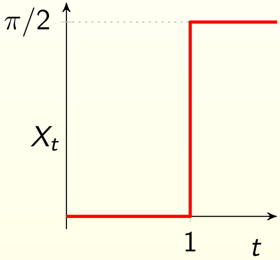
Forward: Y jumps from Y_{t-} to Y_t in the direction of vector field at Y_{t-}.

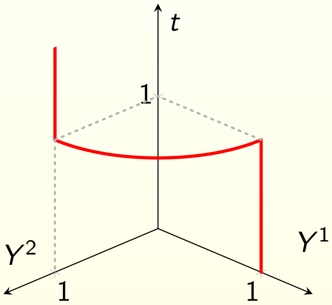


Consider $X_t = \frac{\pi}{2}\mathbf{1}_{t\geq 1}$ and the ODE

$$\begin{pmatrix} \mathrm{d}Y^1 \\ \mathrm{d}Y^2 \end{pmatrix} = \begin{pmatrix} -Y^2 \\ Y^1 \end{pmatrix} \diamond \mathrm{d}X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \diamond \mathrm{d}X \, , \quad \begin{pmatrix} Y^1_0 \\ Y^2_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, .$$

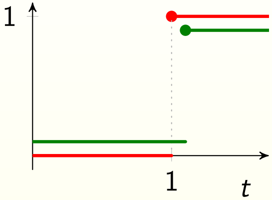
Geometric: Y jumps from Y_{t-} to Y_t along integral curve at Y_{t-}.



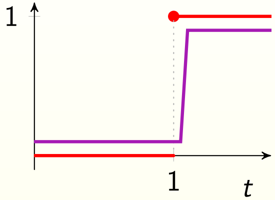


Recap on Skorokhod topologies

J_1: $\sigma_{J_1}(X, Y) = \inf_\lambda \|\lambda - \mathrm{id}\|_\infty + \|X - Y \circ \lambda\|_\infty$



$\sigma_{J_1}(X, Y) \to 0$

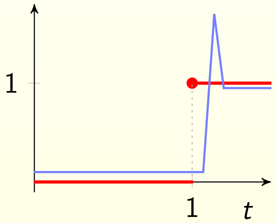


$\sigma_{J_1}(X, Y) \approx 1$

$\sigma_{M_1}(X, Y) \to 0$

M_1: $\sigma_{M_1} \leq \sigma_{J_1}$.

$\sigma_{M_1}(X, Y) \approx \frac{1}{2}$:



J_1-type p-variation topology

Both forward and geometric equations behave well under J_1-type topologies.

- For $p \geq 1$ and $X\colon [0,T] \to \mathbb{R}^d$, denote

$$\|X\|_{p\text{-var}} = \sup_{n\geq 1} \sup_{0\leq t_0 < \ldots < t_n \leq T} \Big(\sum_{i=1}^{n} \|X_{t_i} - X_{t_{i-1}}\|^p \Big)^{1/p}$$

- Define

$$D^{p\text{-var}}([0,T],\mathbb{R}^d) = \{X \in D([0,T],\mathbb{R}^d) \,:\, \|X\|_{p\text{-var}} < \infty\}$$

and the J_1-type p-variation metric

$$\sigma_{p\text{-var}}(X,\bar{X}) = \inf_{\lambda} \|\lambda - \mathrm{id}\|_\infty + \|X - \bar{X}\circ\lambda\|_{p\text{-var}} \,.$$

where the inf is over all increasing bijections $\lambda\colon [0,T] \to [0,T]$.

Stability of ODEs in Young regime:[1]

Theorem

Let $p \in [1,2)$, $b\colon \mathbb{R}^m \to \mathbf{L}(\mathbb{R}^d, \mathbb{R}^m)$ *sufficiently nice, and* $y \in \mathbb{R}^m$. *Then the solution map*

$$D^{p\text{-var}}([0,T], \mathbb{R}^d) \ni X \mapsto Y \in D^{p\text{-var}}([0,T], \mathbb{R}^m)$$

is locally Lipschitz under $\sigma_{p\text{-var}}$, *where* Y *solves*

$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}X_t\ , \quad Y_0 = y \in \mathbb{R}^m\ .$$

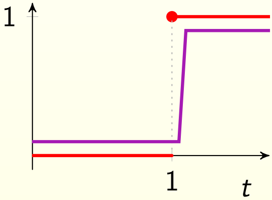
The same holds true if Y *solves the geometric equation*

$$\mathrm{d}Y_t = b(Y_t) \diamond \mathrm{d}X_t\ , \quad Y_0 = y \in \mathbb{R}^m\ .$$

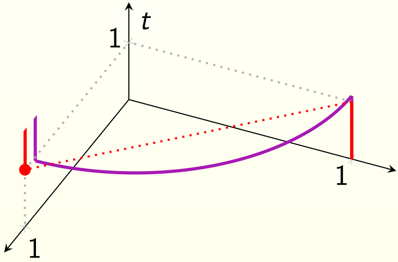
[1] Terry Lyons. *Math. Res. Lett.* (1994); D. R. E. Williams. PhD thesis. Imperial College of London, 1998; Peter K. Friz and Huilin Zhang. *J. Differential Equations* (2018).

Failure for M_1 topology

Neither forward nor geometric equations behave well under M_1-type topologies.



$\sigma_{M_1}(X, \bar{X}) \to 0$



$\sigma_{M_1}(Y, \bar{Y}) \approx \frac{3}{10}$

Generalisation of Skorokhod M_1 topology

Trouble with J_1: continuous paths are closed, so can't study

- mollifications, piecewise linear interpolations, etc.
- small jumps accumulating to a large jump
  - will see this in fast-slow systems later.

Aim: generalise M_1 topology to restore stability for ODEs.

Based on three papers:

- I.C. "Random walks and Lévy processes as rough paths." Probab. Theory Related Fields. (2018) arXiv:1510.09066
- I.C., P. Friz. "Canonical RDEs and general semimartingales as rough paths." Ann. Probab. (2019) arXiv:1704.08053
- I.C., P. Friz, A. Korepanov, I. Melbourne. "Superdiffusive limits for deterministic fast-slow dynamical systems." Probab. Theory Related Fields. (2020) arXiv:1907.04825

Path functions

Definition (Path function)

A *path function* is a map $\phi\colon [0,T] \to C([0,1],\mathbb{R}^d)$ such that

- $\phi_t \equiv \mathrm{const}$ for all but countably many $t_1, t_2, \ldots \in [0,T]$,
- the path $h\colon t \mapsto \phi_t(1)$ is càdlàg with $h_{t-} = \phi_t(0) \;\; \forall t \in [0,T]$
- $\lim_{k\to\infty} \sup_{s\in[0,1]} |\phi_{t_k}(s) - \phi_{t_k}(0)| = 0$

Denote by $\mathscr{D}([0,T],\mathbb{R}^d)$ the space of path functions.

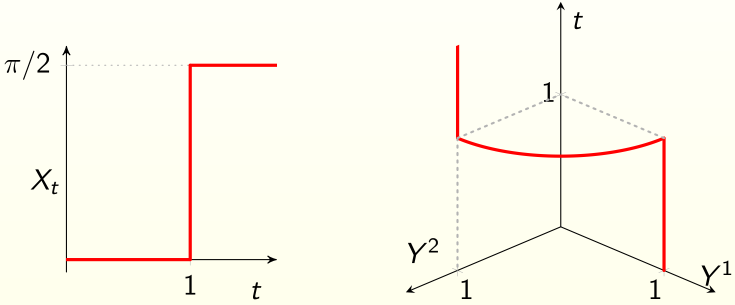
Example (Marcus-like path function)

- Let $h \in D([0,T],\mathbb{R}^d)$ and define $\phi_t \equiv [h_{t-}, h_t]$, where $[x,y]$ is linear path from x to y. Then $\phi \in \mathscr{D}$.

Remark: similar definition appeared for the space 'F' of Whitt.[a]

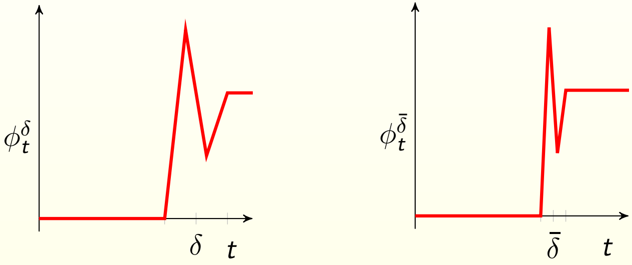
[a] Ward Whitt. *Stochastic-process limits*. Springer-Verlag, 2002, Sec. 15.7.

The point: $\mathscr{D}$ allows us to work with below objects as if they were paths.



p-variation metric on path functions

Each $\phi \in \mathscr{D}$ gives rise (non-uniquely!) to a family $(\phi^\delta)_{\delta>0}$ of continuous functions $\phi^\delta\colon [0, T+\delta] \to \mathbb{R}^d$.



NB. ϕ^δ and $\phi^{\bar\delta}$ are reparametrisations of each other $\forall \delta, \bar\delta > 0$.

p-variation metric on path functions

More precisely: fix $r_k > 0$ with $\sum_{k=1}^{\infty} r_k < \infty$. Let $\{t_i\}_{i=1}^{m}$ be the non-stationary times of ϕ. Let $r = \sum_{k=1}^{m} r_k$ and define

$$\tau_\delta \colon [0, T] \to [0, T + \delta] , \quad \tau_\delta(t) = t + \sum_{k=1}^{m} \frac{\delta r_k}{r} \mathbf{1}_{t_k \le x} .$$

Then define $\phi^\delta \colon [0, T + \delta] \to \mathbb{R}^d$ by

$$\phi^\delta(s) = \begin{cases} \phi_t(1) & \text{if } s = \tau_\delta(t) \text{ for some } t \in [0, T] \\ \phi_{t_k}\left(\frac{s - \tau_\delta(t_k-)}{\delta r_k / r}\right) & \text{if } s \in [\tau_\delta(t_k-), \tau_\delta(t_k)) \text{ for some } 1 \le k < m + 1 . \end{cases}$$

p-variation metric on path functions

Definition

For $p \geq 1$, denote $\mathscr{D}^{p\text{-var}}([0,T],\mathbb{R}^d) = \{\phi \in \mathscr{D} : \|\phi^\delta\|_{p\text{-var}} < \infty\}$.

Lemma/Definition (p-variation metric)

For $p \geq 1$ and $\phi, \bar{\phi} \in \mathscr{D}^{p\text{-var}}([0,T],\mathbb{R}^d)$ the limit

$$\alpha_{p\text{-var}}(\phi,\bar{\phi}) := \lim_{\delta\to 0} \inf_{\lambda} \left[\|\lambda - \mathrm{id}\|_\infty + \|\phi^\delta - \bar{\phi}^\delta \circ \lambda\|_{p\text{-var}} \right],$$

where the inf *is over all increasing bijections $\lambda\colon [0,T+\delta] \to [0,T+\delta]$,*

- *exists,*
- *is independent of r_k,*
- *defines a (pseudo)metric on $\mathscr{D}^{p\text{-var}}$.*

Definition (Geometric ODEs)

For $p \in [1, 2)$, $\phi \in \mathscr{D}^{p\text{-var}}([0, T], \mathbb{R}^d)$, and 'nice' b, we say that $\psi \in \mathscr{D}^{p\text{-var}}([0, T], \mathbb{R}^m)$ solves

$$\mathrm{d}\psi_t = b(\psi_t) \diamond \mathrm{d}\phi_t \ , \quad \psi_0 = y \in \mathbb{R}^m \ ,$$

if ψ^δ solves $\mathrm{d}\psi_t^\delta = b(\psi_t^\delta)\,\mathrm{d}\phi_t^\delta$, $\psi_0^\delta = y$.

Proposition (Stability in Young regime, C.–Friz '19)

For $p \in [1, 2)$ the solution map

$$\mathscr{D}^{p\text{-var}}([0, T], \mathbb{R}^d) \ni \phi \mapsto \psi \in \mathscr{D}^{p\text{-var}}([0, T], \mathbb{R}^m)$$

is locally Lipschitz for $\alpha_{p\text{-var}}$.

Proof: continuous Young ODE theory + careful definition chasing.

Remark: the same definitions, theorems and proofs extend to rough path space by replacing:

- $\mathbb{R}^d \rightsquigarrow G^N(\mathbb{R}^d)$
- $p \in [1,2) \rightsquigarrow p \in [N, N+1)$.

Especially powerful when combined with enhanced p-variation BDG inequality: for a local martingale $X\colon [0,T] \to \mathbb{R}^d$, let $\mathbf{X} = \exp(X + A)$ denote its canonical lift with Lévy area

$$A_t^{i,j} = \frac{1}{2}\int_0^t X_{r-}^i dX_r^j - X_{r-}^j dX_r^i \,.$$

Theorem (C.-Friz '19)

For every convex moderate function F and $p > 2$ there exists $c, C > 0$ such that for every local martingale $X\colon [0,T] \to \mathbb{R}^d$

$$c\mathbb{E}\left[F([X]_\infty^{1/2})\right] \le \mathbb{E}\left[F(||\mathbf{X}||_{p\text{-var}})\right] \le C\mathbb{E}\left[F([X]_\infty^{1/2})\right].$$

Applications to Marcus SDEs

Example applications

Rough path stability + p-var BDG inequality leads to surprising corollaries.

Theorem (Stability under UCV for Marcus SDEs)

Suppose $(X^n)_{n\geq 1}$ is a sequence of semimartingales satisfying UCV and $X^n \to X$ in law (resp. in probability) for the J_1 Skorokhod topology. Then the solutions to the Marcus SDEs

$$\mathrm{d}Y_t^n = V(Y_t^n) \diamond \mathrm{d}X_t^n, \quad Y_0 \in \mathbb{R}^e,$$

converge in law (resp. in probability) for the Skorokhod topology to the solution of the Marcus SDE

$$\mathrm{d}Y_t = V(Y_t) \diamond \mathrm{d}X_t, \quad Y_0 \in \mathbb{R}^e.$$

Kurtz–Protter '91[2] proved this for "Itô" SDEs.

[2] Thomas G. Kurtz and Philip Protter. "Weak limit theorems for stochastic integrals and stochastic differential equations". *Ann. Probab.* (1991).

Example applications

Theorem (Wong–Zakai theorem à la Kurtz–Pardoux–Protter[3])

Let $X\colon [0,T] \to \mathbb{R}^d$ be a semimartingale and $D_n \subset [0,T]$ a sequence of partitions with $\lim_{n\to\infty} |D_n| = 0$. Let X^n be piecewise linear interpolation of X over D_n. Then solutions to the ODEs

$$\mathrm{d}Y_t^n = V(Y_t^n) \diamond \mathrm{d}X_t^n, \quad Y_0 \in \mathbb{R}^e ,$$

converge in $\mathscr{D}$ to ψ in probability. and $Y_t := \psi_t(1)$ solves the Marcus SDE

$$\mathrm{d}Y_t = V(Y_t) \diamond \mathrm{d}X_t, \quad Y_0 \in \mathbb{R}^e .$$

- While $X^n \to X$ in (D, σ_{M_1}), $Y^n \nrightarrow Y$ in (D, σ_{M_1}).
- use of path functions avoids annoyances like "$Y_n(t) \to Y(t)$ for all but countably many $t > 0$" cf. KPP '95.[3]

[3] Thomas G. Kurtz, Étienne Pardoux, and Philip Protter. "Stratonovich stochastic differential equations driven by general semimartingales". *AIHP Prob. Stat.* (1995).

Homogenisation of superdiffusive fast-slow systems

For $n \geq 1$, consider the discrete-time **fast-slow system** on $\mathbb{R}^m \times [0,1]$

$$\begin{cases} y_{j+1}^{(n)} = y_j^{(n)} + n^{-\alpha} b(y_j^{(n)}) v(x_j) & \text{(slow)} \\ x_{j+1} = f(x_j) & \text{(fast)} \end{cases} \qquad (y_0^{(n)}, x_0) \in \mathbb{R}^m \times [0,1] \,,$$

where

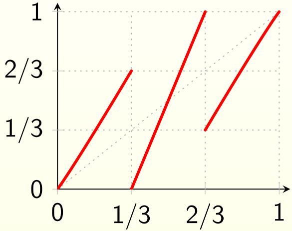
- $y_0^{(n)} \equiv y_0 \in \mathbb{R}^m$ is fixed,
- x_0 is sampled from a probability measure $\mathbb{P}$ on $[0,1]$,
- $v\colon [0,1]\colon\ Y \to \mathbb{R}^d$ — observable
- $b\colon \mathbb{R}^m \to \mathbf{L}(\mathbb{R}^d, \mathbb{R}^m)$ — sufficiently smooth vector field
- $f\colon [0,1] \to [0,1]$ — intermittent map (next slide)

Main question: does the process $Y_n := y_{\lfloor n \cdot \rfloor}^{(n)}$ converge in law as $n \to \infty$?

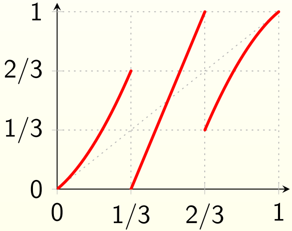
Intermittent map of Pomeau–Manneville type: consider $\gamma \in [0,1)$ and $f\colon [0,1] \to [0,1]$

$$f(x) := \begin{cases} x(1+3^\gamma x^\gamma)\ , & x \in [0, \frac{1}{3})\ , \\ 3x-1\ , & x \in [\frac{1}{3}, \frac{2}{3})\ , \\ 1-(1-x)(1+3^\gamma(1-x)^\gamma)\ , & x \in [\frac{2}{3}, 1]\ . \end{cases}$$

$\gamma = 0.1$:



$\gamma = 0.9$:



Theorem (Ergodicity, Liverani–Saussol–Vaienti '99)

For all $\gamma \in [0,1)$ there exists a unique absolutely continuous, f-invariant, ergodic probability measure $\mathbb{P}$ on $[0,1]$.

Fast-slow systems: ODE formulation

Rewrite the slow dynamic as

$$\begin{cases} \mathrm{d}Y_n(t) & = b(Y_n(t-))\,\mathrm{d}X_n(t)\,, \\ Y_n(0) & = y \in \mathbb{R}^m\,, \end{cases}$$

where

$$X_n(t) = n^{-\alpha} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(x_j)$$

The key is a good understanding of $X_n \colon [0,1] \to \mathbb{R}^d$ as driver.

Diffusive regime: $\gamma < \frac{1}{2}$

For $\gamma < \frac{1}{2}$, take $\alpha = \frac{1}{2}$.

- Melbourne–Nicol '05 CMP: $X_n \xrightarrow{\text{law}} \text{B.m.}$ (in J_1 topology).

For the fast-slow system:

- Kelly–Melbourne '16 AoP: convergence of Y_n to SDE for $\gamma \in [0, \frac{1}{4})$.
  - Uses Hölder rough path topologies
- Kelly–Melbourne '17 JFA: for $\gamma \in [0, \frac{1}{4})$ generalised to non-product systems in continuous-time $\mathrm{d}y_t^{(\varepsilon)} = \varepsilon b(y_t^{(\varepsilon)}, x_t)\,\mathrm{d}t$, $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t$
  - Infinite dimensional Hölder rough paths
  - Bailleul–Catellier '17 JDE: related results using rough flows
- C.–Friz–Korepanov–Melbourne–Zhang '21+ AIHP: discrete- and continuous-time non-product systems.
  - Infinite dimensional p-variation càdlàg rough paths
  - Optimal moment assumptions: handles all $\gamma \in [0, \frac{1}{2})$

Superdiffusive regime: $\gamma > \frac{1}{2}$

Theorem

Let $\gamma \in (\frac{1}{2}, 1)$, $y \sim \mathbb{P}$, *and* $v \in C^{\kappa}([0,1], \mathbb{R})$, $\kappa > 0$, $\int_{[0,1]} v \, \mathrm{d}\mathbb{P} = 0$.

- *(CLT) Gouëzel '04:*[a]

$$X_n(1) = n^{-\gamma} \sum_{j=0}^{n-1} v(x_j) \xrightarrow{\text{law}} G \quad \text{as} \quad n \to \infty \, ,$$

where G *is* $\frac{1}{\gamma}$*-stable law depending only on* $v(0), v(1)$.

- *(Functional CLT) Melbourne–Zweimüller '15:*[b]

$$X_n \xrightarrow{\text{law}} L \quad \text{in} \quad (D([0,1], \mathbb{R}), \sigma_{M_1}) \quad \text{as} \quad n \to \infty \, ,$$

where $L \in D([0,1], \mathbb{R})$ *is* $\frac{1}{\gamma}$*-stable Lévy process.*

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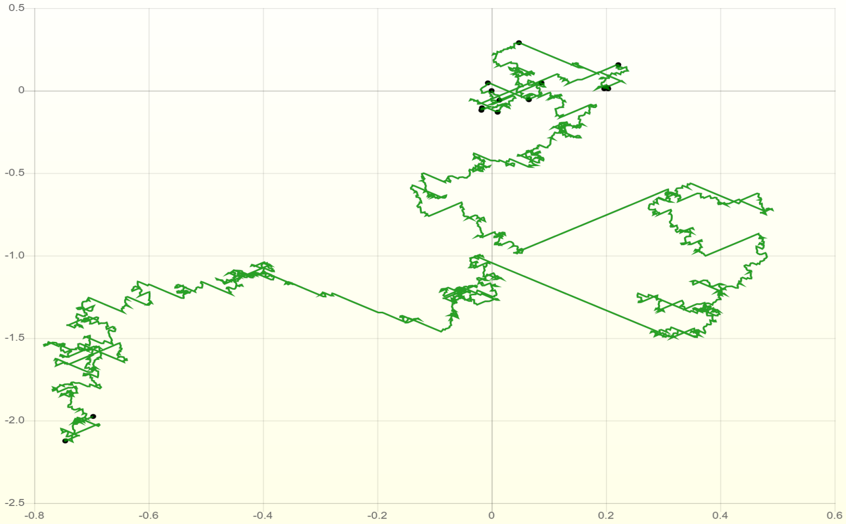
[a] S. Gouëzel. "Central limit theorem and stable laws for intermittent maps". *Probab. Theory Related Fields* (2004).

[b] I. Melbourne and R. Zweimüller. "Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems". *AIHP Prob. Stat.*

Typical realisation of X_n for $v\colon [0,1] \to \mathbb{R}^2$:



Fast-slow system: superdiffusive regime

- Gottwald–Melbourne '13:[4] in case of 1D system, Y_n converges to Marcus SDE.
  - Convergence in M_1 topology.
  - Uses change of coordinates to reduce to additive noise.

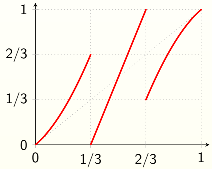
- Difficulties in higher dimensions:
  - Not clear which topology on càdlàg space to use.
  - Requires p-variation estimates on the driver X_n.

[4] G. A. Gottwald and I. Melbourne. "Homogenization for deterministic maps and multiplicative noise". *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* (2013).

In the rest of the talk, we consider:

- Intermittent map $f\colon [0,1] \to [0,1]$ with $\gamma \in (\frac{1}{2}, 1)$:

$$f(x) = \begin{cases} x(1 + 3^\gamma x^\gamma) \ , & x \in [0, \frac{1}{3}) \ , \\ 3x - 1 \ , & x \in [\frac{1}{3}, \frac{2}{3}) \ , \\ 1 - (1-x)(1 + 3^\gamma (1-x)^\gamma) \ , & x \in [\frac{2}{3}, 1] \ . \end{cases}$$



- Unique absolutely continuous, ergodic, f-invariant probability measure $\mathbb{P}$ on $[0,1]$.
- Observable $v \in C^\kappa([0,1], \mathbb{R}^d)$, $\kappa > 0$, $\int_{[0,1]} v \, \mathrm{d}\mathbb{P} = 0$.
- Initial condition $x_0 \sim \mathbb{P}$ and the fast-slow system

$$\begin{cases} y_{j+1}^{(n)} = y_j^{(n)} + n^{-\gamma} b(y_j^{(n)}) v(x_j) & \text{(slow)} \\ x_{j+1} = f(x_j) & \text{(fast)} \end{cases}$$

with sufficiently smooth $b\colon \mathbb{R}^m \to \mathbf{L}(\mathbb{R}^d, \mathbb{R}^m)$ and $y_0 \in \mathbb{R}^m$.

Fast component

Theorem (C.–Friz–Korepanov–Melbourne '20)

1. *The process*

$$X_n(t) = n^{-\gamma} \sum_{j=0}^{\lfloor tn \rfloor - 1} v(x_j)$$

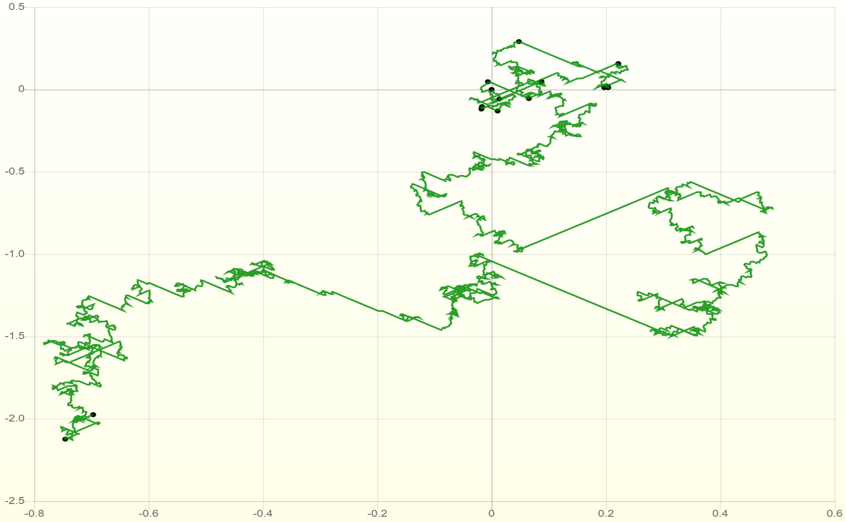
   converges in law to a d-dimensional $\frac{1}{\gamma}$-stable Lévy process in the M_1 topology.

2. *For every $p > \frac{1}{\gamma}$, $\{\|X_n\|_{p\text{-var}}\}_{n\geq 1}$ is tight.*

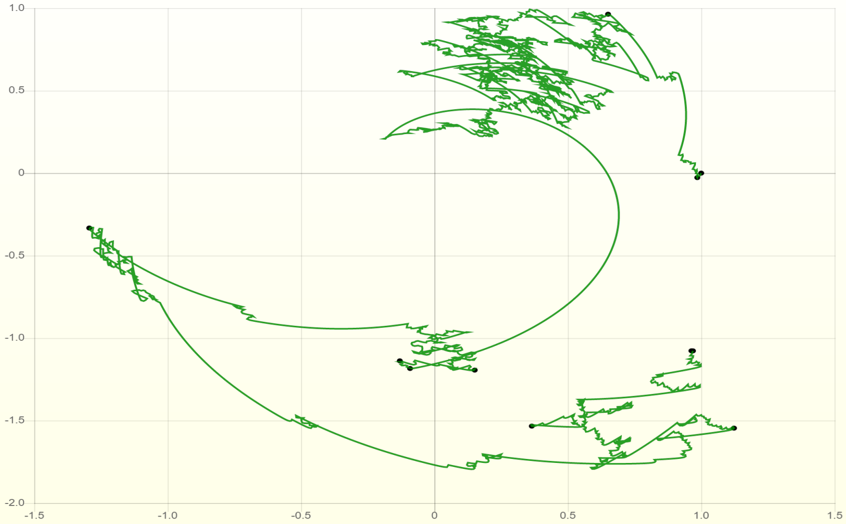
Remarks

- First part generalises Melbourne–Zweimüller '15 to $d \geq 2$
- Uses BDG-type inequality for p-variation bounds

Recall a typical realisation of X_n for $v \colon Y \to \mathbb{R}^2$:



Typical realisation of $Y_n = y^{(n)}_{\lfloor n \cdot \rfloor}$ (clearly does not converge in M_1).



Theorem (C.–Friz–Korepanov–Melbourne '20)

For $p > \frac{1}{\gamma}$,

$$Y_n \to \psi \quad \textit{as} \quad n \to \infty \quad \textit{in law in } (\mathscr{D}^{p\text{-var}}([0,T],\mathbb{R}^m), \alpha_{p\text{-var}}) \; .$$

Moreover, $Y_t := \psi_t(1)$ solves the Marcus SDE

$$\mathrm{d}Y = b(Y) \diamond \mathrm{d}L \; ,$$

where L is d-dimensional $\frac{1}{\gamma}$-stable Lévy process.

Corollary

$Y_n \to Y$ in the sense of finite dimensional distributions.

Can readily generalise to:

- class of nonuniformly expanding fast dynamics
- drift term $n^{-1}a(y_j^{(n)})$ in slow dynamics
- initial condition $x_0 \sim \mathbb{Q}$ for any $\mathbb{Q} \ll \mathbb{P}$.

Summary

- Introduced a space $\mathscr{D}$ to make sense of instantaneous continuous movement.
  - Convenient for analysis.
  - Combined with rough paths $\Rightarrow$ applications to classical Marcus SDEs.
- Young case $\Rightarrow$ homogenisation of superdiffusive fast-slow systems.
- Future directions:
  - Billiards – here X_n does not converge in M_1 $\Rightarrow$ need path functions for drivers too, not Marcus SDE in the limit.
  - Fast-slow systems with non-product structure

$$\begin{cases} y_{j+1}^{(n)} = y_j^{(n)} + n^{-\alpha} b(y_j^{(n)}, x_j) \\ x_{j+1} = f(x_j) \end{cases}$$

  - Coupling between fast and slow component (very difficult...).

Thank you!