Tempered representations of real semisimple Lie groups
3. Classification of tempered representations

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So far

We have seen that

- the **tempered representations** of G are the irreducible unitary representations that occur in the Plancherel decomposition of $L^2(G)$
- the direct summands in this decomposition are the **discrete series** representations; they exist if G has a compact Cartan and can then be classified
- a **cuspidal parabolic subgroup** of G is of the form $P = MAN$, where
  - M is 'like G', but may be disconnected
  - $A \cong \mathbb{R}^n$
  - the Lie algebra of N is nilpotent
- unitary representations $\sigma \in \hat{M}$ and $\nu \in \hat{A}$ allow us to define the unitary **induced representation** $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ of G.

Today

Today, we combine the theory from the first two lectures to classify almost all tempered representations of G.

Relevant questions:

1. Can all tempered representations of G be constructed from/found inside induced representations?
2. Which induced representations are equivalent?
3. Which induced representations are irreducible?

The second and third questions can be answered via **intertwining operators**.

I Intertwining operators

Schur's lemma

For any two unitary representation π, π' of G, consider the space

$$\mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})^G := \{T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'}); \forall g \in G, \pi'(g)T = T\pi(g)\}$$

of **intertwining** bounded operators. We write

$$\mathcal{B}(\mathcal{H}_\pi)^G = \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\pi)^G,$$

which is an algebra.

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Lemma (Schur)

(a) *If* $\pi \in \hat{G}$, *then*

$$\mathcal{B}(\mathcal{H}_\pi)^G = \mathbb{C}\,\mathrm{Id}\,.$$

(b) *If also* $\pi' \in \hat{G}$, *then*

$$\dim \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})^G = \begin{cases} 0 & \text{if } \pi \not\cong \pi' \\ 1 & \text{if } \pi \cong \pi'. \end{cases}$$

Idea of proof

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Proof.

(a) If $\mathcal{H}_\pi$ is finite-dimensional and $T \in \mathcal{B}(\mathcal{H}_\pi)^G$ is self-adjoint, then every eigenspace of T is an invariant subspace of $\mathcal{H}_\pi$. So there is only one eigenspace.

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In general, use decomposition of T into self-adjoint operators, and spectral projections.

(b) If $T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})^G$, then $\ker(T) \subset \mathcal{H}_\pi$ and $\mathrm{im}(T) \subset \mathcal{H}_{\pi'}$ are invariant subspaces.

□

A consequence of Schur's lemma

Let $\pi_1, \ldots, \pi_n \in \hat{G}$ be mutually inequivalent. Let

$$\pi = \bigoplus_{j=1}^{n} m_j \pi_j,$$

where $m_j \pi_j$ is the direct sum of m_j copies of π_j. Then

$$\mathcal{B}(\mathcal{H}_\pi)^G \cong \bigoplus_{j=1}^{n} M_{m_j}(\mathbb{C}).$$

Moral: if we know $\mathcal{B}(\mathcal{H}_\pi)^G$, and it is finite-dimensional, then we know how π decomposes into irreducibles.

Action by $N_K(\mathfrak{a})$ on representations

Let $P = MAN$ be a cuspidal parabolic subgroup, $\sigma \in \hat{M}$. Any element $\xi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$ defines a representation ν of A by

$$\nu(\exp(X)) = e^{\langle \xi, X \rangle}.$$

This representation is unitary if ξ is imaginary-valued.

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Let

$$w \in N_K(\mathfrak{a}) = N_K(A) = \{k \in K; kAk^{-1} = A\}.$$

Then conjugation by w preserves $Z_{\mathfrak{g}}(\mathfrak{a})$, and hence $\mathfrak{m}$ and M. So we can define representations

$$(w \cdot \sigma)(m) := \sigma(w^{-1}mw)$$
$$(w \cdot \nu)(a) := \nu(w^{-1}aw),$$

of M and A, respectively.

A Weyl group

The group

$$W := N_K(\mathfrak{a})/(K \cap M)$$

is finite.

The action by $N_K(\mathfrak{a})$ on $\hat{M} \times \hat{A}$ descends to an action by W: for all $k \in K \cap M$, $\sigma \in \hat{M}$ and $\nu \in \hat{A}$,

- $k \cdot \sigma \cong \sigma$, because they differ by an inner automorphism
- $k \cdot \nu = \nu$, because M acts trivially on A.

Intertwining operators

Let

$$\bar{N} := \{(n^*)^{-1}; n \in N\}.$$

If $F \in \mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N)$, $w \in N_K(\mathfrak{a})$, and $g \in G$ are such that the integral converges, define

$$(A_P(w, \sigma, \nu)F)(g) = \int_{\bar{N} \cap wNw^{-1}} F(gw\bar{n})\, d\bar{n}.$$

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Theorem (Knapp–Stein, 1980)

The integral converges for F in a dense subspace of $\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N)$ if $\mathrm{Re}(\xi)$ *has large enough inner products with a certain subset of Σ^+.*

This extends meromorphically in ξ.

To cancel poles in the meromorphic continuation, we sometimes need **normalised intertwining operators** $\mathcal{A}_P(w, \sigma, \nu)$, equal to $A_P(w, \sigma, \nu)$ times a meromorphic function in ξ.

Different N

There are similar intertwining operators that relate representations of the forms $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes \nu \otimes 1_N)$ and $\mathrm{Ind}_{MAN'}^{G}(\sigma \otimes \nu \otimes 1_{N'})$ to each other.

Intertwining and cocycle properties

Theorem (Knapp–Stein, 1980)

For all $w \in N_K(\mathfrak{a})$ and $\nu \in \hat{A}$, the operator $\mathcal{A}_P(w, \sigma, \nu)$ is a unitary intertwining operator

$$\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N) \to \mathcal{H}_P^G(w\sigma \otimes w\nu \otimes 1_N).$$

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Theorem (Knapp–Stein, 1980)

For all $w, w' \in N_K(\mathfrak{a})$,

$$\mathcal{A}_P(ww', \sigma, \nu) = \mathcal{A}_P(w, w'\sigma, w'\nu)\mathcal{A}_P(w', \sigma, \nu).$$

Elements fixing σ

Lemma

Let $w \in N_K(\mathfrak{a})$, and suppose that $w\sigma \cong \sigma$. Then the representation σ extends to the group generated by M and w. And

$$\sigma(w)\colon \mathcal{H}_\sigma \to \mathcal{H}_\sigma$$

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Corollary

If $w \in N_K(\mathfrak{a})$ and $w\sigma \cong \sigma$ and $w\nu = \nu$, then we have the unitary intertwining operator

$$\sigma(w)\mathcal{A}_P(w,\sigma,\nu)\colon \mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N) \to \mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N).$$

Harish-Chandra's completeness theorem

Let $P = MAN < G$ be a cuspidal parabolic subgroup. Let $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$. Consider the finite group

$$W_{\sigma,\nu} := \{w \in N_K(\mathfrak{a}); w\sigma \cong \sigma, w\nu = \nu\}/(K \cap M) < W.$$

Theorem (Harish-Chandra 1976, Knapp–Stein 1980)

$$\mathcal{B}(\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N))^G = \mathrm{span}\,\{\sigma(w)\mathcal{A}_P(w, \sigma, \nu), w \in W_{\sigma,\nu}\}\,.$$

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Corollary

For all $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$,

$$\dim \mathcal{B}(\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N))^G \leq \# W_{\sigma,\nu}.$$

So if $W_{\sigma,\nu} = \{e\}$, then $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible.

Irreducibility for regular ν

A representation $\nu \in \hat{A}$ is **regular** if $w\nu \neq \nu$ for all $w \in W \setminus \{e\}$. The regular irreducible representations form an open dense subset of $\hat{A} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, i\mathbb{R})$.

Corollary

If $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$ is regular, then $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible.

Proof.

If ν is regular, then $W_{\sigma,\nu} = \{e\}$. □

Example: the discrete series

As an extreme case, suppose that G has a compact Cartan subgroup, and let $\pi_\lambda \in \hat{G}_{\text{discr}}$. Take the cuspidal parabolic subgroup $P = G$. Then $M = G$ and $A = N = \{e\}$. And

$$\mathrm{Ind}_G^G(\pi_\lambda \otimes 1_A \otimes 1_N) = \pi_\lambda.$$

Now $\mathfrak{a} = \{0\}$, so

$$W = N_K(\mathfrak{a})/(K \cap M) = K/K = \{e\}.$$

Hence $W_{\sigma,1_A}$ is trivial, which is consistent with irreducibility of π_λ.

Example: the principal series of $\mathrm{SL}(2, \mathbb{R})$

Consider the cuspidal parabolic subgroup $P = MAN < \mathrm{SL}(2, \mathbb{R})$ with

$$M = \{\pm I\}$$
$$A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} ; r > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ; x \in \mathbb{R} \right\} \cong \mathbb{R}$$

Let σ_+ be the trivial representation of M, and σ_- the nontrivial irreducible one. Let $\nu \in \hat{A} \cong \mathbb{R}$.

Weyl group

Now

$$N_K(\mathfrak{a}) = \left\{ k \in \mathrm{SO}(2); k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} k^{-1} \in \mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$= \left\{ \pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

And $K \cap M = \{\pm I\}$, so

$$W = N_K(\mathfrak{a})/(K \cap M) = \left\{ e, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M \right\}.$$

The action by the nontrivial element on $\mathfrak{a} \cong \mathbb{R}$ is reflection in 0.

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Now W acts trivially on M, so

$$W_{\sigma_\pm, \nu} = \begin{cases} \{e\} & \text{if } \nu \neq 0 \\ W & \text{if } \nu = 0. \end{cases}$$

We find that $\mathrm{Ind}_P^G(\sigma_\pm \otimes \nu \otimes 1_N)$ is irreducible if $\nu \neq 0$.

(Ir)reducibility

We saw that if $\nu = 0 = 1_A$,

$$W_{\sigma_\pm, 1_A} = W.$$

So

$$\dim \left(\left(\mathcal{B}(\mathcal{H}_P^G(\sigma_\pm \otimes 1_A \otimes 1_N) \right) \right)^G \leq \# W = 2.$$

So $\mathrm{Ind}_P^G(\sigma_\pm \otimes 1_A \otimes 1_N)$ decomposes into **at most two** irreducible tempered representations.

Intertwining operators

Let $w \in W$ be the nontrivial element. Then for all $F_\pm \in \mathcal{H}_P^G(\sigma_\pm \otimes \nu \otimes 1_N) \cong L^2(\mathbb{R})$,

$$(A_P(w, \sigma_+, i\nu)F_+)(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{F_+(x-y)}{|y|^{1-i\nu-t}}\, dy$$

and

$$(A_P(w, \sigma_-, i\nu)F_-)(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{F_-(x-y)\,\mathrm{sign}(y)}{|y|^{1-i\nu-t}}\, dy.$$

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These implement the only equivalences between the representations $\mathrm{Ind}_P^G(\sigma_\pm \otimes \nu \otimes 1_N)$:

$$\mathrm{Ind}_P^G(\sigma_\pm \otimes \nu \otimes 1_N) \cong \mathrm{Ind}_P^G(\sigma_\pm \otimes -\nu \otimes 1_N).$$

II Classification of almost all tempered representations

Exhaustion

We have seen that for all $\sigma \in \hat{M}_{\mathrm{discr}}$ and almost all $\nu \in \hat{A}$, the induced representation $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible.

Theorem

Let $\sigma \in \hat{M}$ be tempered and $\nu \in \hat{A}$. Then each irreducible summand of $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is a tempered representation of G.

So $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is an (irreducible) tempered representation for regular ν.

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Theorem (Harish-Chandra, Langlands, Trombi)

Every tempered representation of G is contained in $\text{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$, for some cuspidal parabolic P, $\sigma \in \hat{M}_{\text{discr}}$ and $\nu \in \hat{A}$.

Langlands disjointness theorem

Theorem (Harish-Chandra, Langlands)

For $j = 1, 2$, consider cuspidal parabolics $P_j = M_j A_j N_j$, and $\sigma_j \in \hat{M}_j$ and $\nu_j \in \hat{A}_j$. If $\mathrm{Ind}_{P_1}^G(\sigma_1 \otimes \nu_1 \otimes 1_N)$ and $\mathrm{Ind}_{P_2}^G(\sigma_2 \otimes \nu_2 \otimes 2)$ have a common irreducible constituent, then there is a $k \in K$ such that

$$M_2 = kM_1k^{-1}$$
$$A_2 = kA_1k^{-1}$$
$$\sigma_2 = k\sigma_1$$
$$\nu_2 = k\nu_1.$$

The classification

Theorem

The representations $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$, *where*

1. $P = MAN < G$ *runs over the finite set of cuspidal parabolics coming from representatives of each conjugacy class of Cartan subgroups*
2. $\sigma \in \hat{M}_{\mathrm{discr}}$
3. $\nu \in \hat{A}$ *is regular*

are irreducible and tempered. Almost every irreducible tempered representation of G *is of this form.*

Two such representations $\mathrm{Ind}_{P_1}^G(\sigma_1 \otimes \nu_1 \otimes 1_N)$ *and* $\mathrm{Ind}_{P_2}^G(\sigma_2 \otimes \nu_2 \otimes 1_N)$ *are equivalent if and only if* $P_1 = P_2$, *and there is a* $w \in N_K(\mathfrak{a})/(K \cap M)$ *such that* $\sigma_2 = w\sigma_1$ *and* $\nu_2 = w\nu_1$.

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The "almost every" claim follows from the fact that the Plancherel measure is a function times Lebesgue measure on $\hat{A} = \mathbb{R}^n$, so almost all $\nu \in \hat{A}$ are regular.

Example: complex semisimple groups

If $G = \mathrm{SL}(n, \mathbb{C})$, or any other complex semisimple Lie group, then

- all its Cartan subgroups are conjugate, and so are all its cuspidal parabolic subgroups
- all parabolically induced representations (the principal series) are irreducible.

So

$$\hat{G}_{\mathrm{temp}} = (\hat{M} \times \hat{A})/W.$$

And M is a torus, of the same dimension as A. So

$$\hat{M} = \mathbb{Z}^n \qquad \hat{A} = \mathbb{R}^n.$$

Example: $\mathrm{SL}(2, \mathbb{R})$

The tempered representations of $\mathrm{SL}(2, \mathbb{R})$, up to a set of Placherel measure zero, are

- the **discrete series** representations D_n^+ and D_n^- for $n \in \mathbb{N}$, for $P = G$
- the **principal series** representations $\mathrm{Ind}_P^G(\sigma_+ \otimes \nu \otimes 1_N)$ and $\mathrm{Ind}_P^G(\sigma_- \otimes \nu \otimes 1_N)$, for $\nu \in \hat{A} = \mathbb{R}$ positive, for P upper triangular.

All of these are irreducible, none are equivalent.

III Reducibility of induced representations

Reducibility

We have seen that $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible for $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$ regular. In the rest of this lecture, we investigate the reducibility for singular ν, especially $\nu = 0$.

This is motivated by

- the classification of all tempered representations by Knapp and Zuckermann in 1982
- the description of $C_r^*(G)$ and its K-theory as in Bram's last lecture.

A key concept is the R-**group**.

R-groups

We saw that the operators $\sigma(w)\mathcal{A}_P(w, \sigma, \nu)$, for $w \in W_{\sigma,\nu}$, span $\mathcal{B}(\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N))^G$. For more detailed information, we would like to find a **basis** inside this spanning set.

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Suppose that $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$. Consider the subgroup

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Theorem (Knapp–Stein, 1980; Knapp 1982)

There is a unique subgroup $R_{\sigma,\nu} < W_{\sigma,\nu}$ such that

1. $W_{\sigma,\nu} = W'_{\sigma,\nu} \rtimes R_{\sigma,\nu}$
2. $R_{\sigma,\nu} \cong (\mathbb{Z}/2\mathbb{Z})^n$ *for some* $n \leq \dim(A)$
3. *the set of operators*

$$\{\sigma(w)\mathcal{A}_P(w,\sigma,\nu); w \in R_{\sigma,\nu}\}$$

is a basis of $\mathcal{B}(\mathcal{H}_P^G(\sigma\otimes\nu\otimes 1_N))^G$.

Commutativity of the intertwiner algebra

Corollary (Multiplicity one theorem)

For $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$, the algebra $\mathcal{B}(\mathcal{H}_P^G(\sigma, \otimes\nu \otimes 1_N))^G$ is commutative.

Proof.

This follows from commutativity of $R_{\sigma,\nu}$ and the cocycle relation

$$\sigma(w)\mathcal{A}_P(w,\sigma,\nu)\sigma(w')\mathcal{A}_P(w',\sigma,\nu) = c\sigma(ww')\mathcal{A}_P(ww',\sigma,\nu),$$

if $w, w' \in N_K(\mathfrak{a})$ fix σ and ν, where

$$\sigma(ww')^{-1}\sigma(w)\sigma(w') = c\,\mathrm{Id}\,.$$

□

Reducibility

Corollary

For all $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\sigma \in \hat{A}$, the representation $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is a direct sum of $\#R_{\sigma,\nu}$ inequivalent irreducible tempered representations.

Proof.

The commutative algebra $\mathcal{B}(\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N))^G$ equals

$$\bigoplus_{j=1}^{k} M_{m_j}(\mathbb{C})$$

if the jth irreducible summand occurs m_j times. So $m_j = 1$ for all j. So

$$k = \sum_{j=1}^{k} m_j^2 = \dim\left(\mathcal{B}(\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N))^G\right) = \#R_{\sigma,\nu}.$$

□

Maximally compact Cartans

Theorem

If P is associated to a **maximally compact Cartan**, *then $R_{\sigma,\nu} = \{e\}$ for all $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$.*

Then $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible.

Maximally compact Cartans

Theorem

If P is associated to a **maximally compact Cartan**, *then $R_{\sigma,\nu} = \{e\}$ for all $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$.*

Then $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is irreducible.

This applies in particular to the discrete series, and to complex semisimple groups.

Example: the spherical principal series of $\mathrm{SL}(2,\mathbb{R})$

Let $P = MAN$ be the minimal cuspidal parabolic in $\mathrm{SL}(2,\mathbb{R})$, of upper-triangular matrices. Let σ_+ be the trivial representation of $M = \{\pm I\}$. The spherical principal series representation $\mathrm{Ind}_P^G(\sigma_+ \otimes 1_A \otimes 1_N)$ is **irreducible**: one can show directly that

$$\mathcal{B}(\mathcal{H}_P^G(\sigma_+ \otimes 1_A \otimes 1_N))^G = \mathbb{C}\,\mathrm{Id}\,.$$

So

$$W'_{\sigma_+,1_A} = \mathbb{Z}/2\mathbb{Z}$$
$$R_{\sigma_+,1_A} = \{e\}.$$

Example: the non-spherical principal series of $\mathrm{SL}(2, \mathbb{R})$

We have

$$L^2(\mathbb{R}) \cong V_+ \oplus V_-,$$

where

- V_+ consists of the restrictions to $\mathbb{R}$ of the continuous extensions of holomorphic functions f on the upper half-plane for which

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < \infty$$

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Let σ_- be the nontrivial irreducible representation of $M = \{\pm I\}$. Then V_+ and V_- are invariant under $\mathrm{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1_N)$.

And we saw that $\mathrm{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1_N)$ decomposes into at most 2 irreducible tempered representations. So

$$W'_{\sigma_-,1_A} = \{e\}$$
$$R_{\sigma_-,1_A} = \mathbb{Z}/2\mathbb{Z}.$$

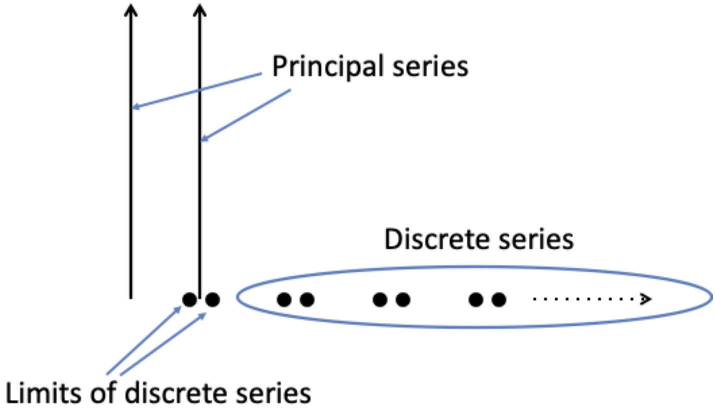
The tempered representations of $\mathrm{SL}(2,\mathbb{R})$

We have now seen all tempered representations of $\mathrm{SL}(2,\mathbb{R})$:

- the **discrete series** representations D_n^+ and D_n^- for $n \in \mathbb{N}$
- the **spherical principal series** representations $\mathrm{Ind}_P^G(\sigma_+ \otimes \nu \otimes 1)$, for $\nu \geq 0$
- the **non-spherical principal series** representations $\mathrm{Ind}_P^G(\sigma_- \otimes \nu \otimes 1)$, for $\nu > 0$
- the **limits of discrete series** $\pi_+ \colon G \to \mathrm{U}(V_+)$ and $\pi_- \colon G \to \mathrm{U}(V_-)$, the irreducible components of $\mathrm{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1)$.

All of these are irreducible, none are equivalent.

The tempered representations of $SL(2, \mathbb{R})$



Classification of all tempered representations

Knapp and Zuckerman gave a complete classification of all tempered representations for all semisimple G in 1982. This involves

- R-groups
- **limits of discrete series** representations
- induced representations with **nondegenerate data**.

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- induced representations with **nondegenerate data**.

Together with the (earlier) Langlands classification, this gives a classification of all irreducible **admissible** representations.

As mentioned in the first lecture, it is in general unknown which admissible representations are/can be made **unitary**.

Notation when ν is trivial

Let $P = MAN$ be a cuspidal parabolic, and $\sigma \in \hat{M}_{\mathrm{discr}}$. We write

$$W_\sigma := W_{\sigma,1_A} \qquad W'_\sigma := W'_{\sigma,1_A} \qquad R_\sigma := R_{\sigma,1_A},$$

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Via the compact picture, every representation $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1)$ can be realised in

$$\mathcal{H}_P^G(\sigma) := \mathcal{H}_P^G(\sigma \otimes 1_A \otimes 1).$$

Group actions in the computation of $C_r^*(G)$

In the description of $C_r^*(G)$, key roles are played by the actions

- by W_σ on $C_0(\hat{A}, \mathcal{K}(\mathcal{H}_P^G(\sigma)))$ by

$$(w \cdot \varphi)(\nu) = \sigma(w)\mathcal{A}_P(\sigma \otimes \xi \otimes 1)\varphi(w^{-1}\nu)\,(\sigma(w)\mathcal{A}_P(\sigma \otimes \xi \otimes 1))^{-1}\,,$$

  for $w \in W_\sigma$, $\varphi \in C_0(\hat{A}, \mathcal{K}(\mathcal{H}_P^G(\sigma)))$ and $\nu \in \hat{A}$

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  for $w \in W_\sigma$, $\varphi \in C_0(\hat{A}, \mathcal{K}(\mathcal{H}_P^G(\sigma)))$ and $\nu \in \hat{A}$
- by R_σ on $C_0(\mathfrak{a}/W'_\sigma, \mathcal{K}(l^2(R_\sigma))$ by

$$(w \cdot \psi)(X) = L_w^{R_\sigma}\psi(\mathrm{Ad}(w)^{-1}X)L_{w^{-1}}^{R_\sigma},$$

  for $w \in R_\sigma$, $\psi \in C_0(\mathfrak{a}/W'_\sigma, \mathcal{K}(l^2(R_\sigma))$ and $X \in \mathfrak{a}/W'_\sigma$, where L^{R_σ} is the left regular representation of R_σ.

The case $W'_\sigma = \{e\}$

Let $A_{\max}$ be the lowest-dimensional A that occurs.

Theorem (Knapp–Zuckerman)

If $W'_\sigma = \{e\}$, then $R_\sigma = (\mathbb{Z}/2)^{\dim(A) - \dim(A_{\max})}$, and $\mathfrak{a}$ is R_σ-equivariantly isomorphic to $\mathbb{R}^{\dim(A)}$, on which R_σ acts by reflections in the first $\dim(A) - \dim(A_{\max})$ coordinates.

Summary

We have seen that

- the tempered representations of G are the irreducible unitary representations that occur in the Plancherel decomposition of $L^2(G)$
- the direct summands in this decomposition are the discrete series representations
- almost all tempered representations of G are of the form

  $$\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1)$$

  for a cuspidal parabolic $P = MAN < G$, $\sigma \in \hat{M}_{\mathrm{discr}}$ and $\nu \in \hat{A}$ regular
- for singular ν, the reducibility of $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1)$ can be described in terms of R-groups.