Tempered representations of real semisimple Lie groups
2. Parabolic induction

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Previous lecture

In the first lecture, we saw that

- the tempered representations of G are the unitary irreducible representations that occur in the Plancherel decomposition of $L^2(G)$
- If G has a compact Cartan subgroup, then its has countably many discrete series representations. These can be classified explicitly.

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In this lecture, we work towards the classification of almost all tempered representations, by

- discussing how a unitary representation of a subgroup of G can be **induced** to a unitary representation of G
- introducing a class of subgroups to which we will apply this induction procedure: **cuspidal parabolic subgroups**.

I Unitary induction

The idea

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Let $\pi \colon H \to \mathrm{U}(\mathcal{H}_\pi)$ be a unitary representation. We would like to construct an **induced representation** of G that is

- unitary
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Geometric idea:

- Construct the (possibly infinite-rank) G-vector bundle

$$G \times_H \mathcal{H}_\pi = \frac{G \times \mathcal{H}_\pi}{(gh^{-1}, \pi(h)v) \sim (g, v)} \to G/H.$$

- Form the space of square integrable sections of this bundle, twisted by "half-densities".
- Take the representation of G in this space coming from left multiplication.

Modular functions

Let dg be a left Haar measure on G. Then for all $f \in C_c(G)$ and all $g \in G$,

$$\int_G f(gg')\, dg' = \int_G f(g')\, dg'.$$

The **modular function** associated to dg is the group homomorphism $\Delta_G \colon G \to (0, \infty)$ such that for all $f \in C_c(G)$ and all $g \in G$,

$$\int_G f(g'g)\, dg' = \Delta_G(g)^{-1} \int_G f(g')\, dg'.$$

If G is Lie group, then for all $g \in G$,

$$\Delta_G(g) = |\det(\mathrm{Ad}(g))|^{-1}.$$

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Similarly, we have a left Haar measure dh and modular function Δ_H on H. We will use the homomorphism

$$\chi := \frac{\Delta_G^{1/2}|_H}{\Delta_H^{1/2}} \colon H \to (0, \infty).$$

An induced Hilbert space

Consider the vector space $C(G, \mathcal{H}_\pi \otimes \mathbb{C}_\chi)^H$ of continuous functions $F\colon G \to \mathcal{H}_\pi$ such that for all $g \in G$ and $h \in H$,

$$F(gh) = \frac{\Delta_G^{1/2}(h)}{\Delta_H^{1/2}(h)} \pi(h)^{-1} F(g).$$

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To define an integral of such functions "over G/H", we choose a function $\varphi \in C_c(G)$ such that for all $g \in G$,

$$\int_H \varphi(gh)\, dh = 1.$$

The inner product of $F_1, F_2 \in C(G, \mathcal{H}_\pi \otimes \mathbb{C}_\chi)^H$ is

$$(F_1, F_2) := \int_G (F_1(g), F_2(g))_{\mathcal{H}_\pi} \varphi(g)\, dg.$$

We then take the completion $\mathcal{H}_H^G(\pi)$ of $C(G, \mathcal{H}_\pi \otimes \mathbb{C}_\chi)^H$ in this inner product.

Unitary induction

Definition

The representation $\mathrm{Ind}_H^G(\pi)$ of G in $\mathcal{H}_H^G(\pi)$ given by

$$\big(\mathrm{Ind}_H^G(\pi)(g)F\big)(g') = F(g^{-1}g')$$

is the **unitary induction** of π from H to G.

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Lemma

The representation $\operatorname{Ind}_H^G(\pi)$ *is unitary.*

Ingredient: if $K < G$ is a compact subgroup such that $G = KH$, then for all $f \in C_c(G)$,

$$\int_G f(g)\,dg = \int_K \int_H f(kh) \frac{\Delta_H(h)}{\Delta_G(h)}\,dh\,dk.$$

Example: $H = G$

Suppose that $H = G$. Then $\chi = 1$, and the evaluation map at e defines a unitary equivalence

$$C(G, \mathcal{H}_\pi)^G \cong \mathcal{H}_\pi$$

of representations of G. So

$$\mathrm{Ind}_G^G(\pi) = \pi.$$

The discrete series of disconnected groups

Even if G is connected, we will use discrete series representations of possibly disconnected subgroups. These can be constructed via unitary induction.

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Suppose for this slide and the next that

- $G < \mathrm{GL}(n, \mathbb{C}) < \mathrm{GL}(2n, \mathbb{R})$ is a closed subgroup, closed under conjugate transpose, with finitely many connected components
- the connected component G_0 of the identity has compact centre
- if $G^{\mathbb{C}} < \mathrm{GL}(2n, \mathbb{C})$ is the subgroup generated by $\exp(\mathfrak{g} \otimes \mathbb{C})$, suppose that

$$G \subset G^{\mathbb{C}} Z_{\mathrm{GL}(n,\mathbb{C})}(G).$$

The discrete series of disconnected groups (cont'd)

Let $T < G$ be a compact Cartan subgroup. Let $\pi_\lambda^{G_0}$ be a discrete series representation of G_0, with corresponding parameter $\lambda \in i\mathfrak{t}^*$. Let $\chi \in \widehat{Z(G)}$ be such that

$$\chi|_{T \cap Z(G)} = e^{\lambda - \rho^\lambda}|_{T \cap Z(G)}.$$

Then we have the well-defined representation $\pi_\lambda^{G_0} \boxtimes \chi$ of $G_0 Z(G)$, given by

$$(\pi_\lambda^{G_0} \boxtimes \chi)(g_0 z) := \pi_\lambda^{G_0}(g_0)\chi(z),$$

for $g_0 \in G_0$ and $z \in Z(G)$. Then we have the discrete series representation

$$\pi_{\lambda,\chi} := \mathrm{Ind}_{G_0 Z(G)}^{G}(\pi_\lambda^{G_0} \boxtimes \chi)$$

of G.

The discrete series of disconnected groups (cont'd)

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of G.

All discrete series representations of G are of this form, and $\pi_{\lambda,\chi}$ is equivalent to $\pi_{\lambda',\chi'}$ if and only if $\chi \cong \chi'$, and there is a $w \in N_G(T)/Z_G(T)$ such that $\lambda' = w\lambda$.

II Cuspidal parabolic subgroups

Cuspidal parabolic subgroups

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- G is closed under conjugate transpose
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- G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$
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- G has finite centre.

To classify the tempered representations of G, we consider **cuspidal parabolic subgroups** $P = MAN$, where

- M is as in the discussion of the discrete series of disconnected groups
- $A \cong \mathbb{R}^n$
- N is nilpotent.

We use unitary induction from P to G of discrete series representations of M, combined with representations of A.

Cartan decomposition

Let $K := G \cap \mathrm{U}(n)$, a **maximal compact subgroup**. Let

$$\mathfrak{s} := \{X \in \mathfrak{g}; X^* = X\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$.

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$$\mathfrak{s} := \{X \in \mathfrak{g}; X^* = X\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$.

Let $H < G$ be a θ-**stable Cartan subgroup**: a Cartan subgroup such that

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{s}).$$

The term θ-stable refers to the **Cartan involution**

$$\theta(X) = -X^*.$$

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Consider the inner product

$$(X, Y) := -\operatorname{Re}\operatorname{tr}(X\theta Y)$$

on $\mathfrak{g}$.

The Levi factor

Define

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- $A := \exp(\mathfrak{a})$.

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Define

- $\mathfrak{m}$ to be the orthogonal complement to $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a})$
- $M_0 < G$ to be the subgroup generated by $\exp(\mathfrak{m})$
- $M := Z_K(\mathfrak{a})M_0$.

Then M satisfies the assumptions in the discussion of the discrete series of disconnected groups, and M has **discrete series representations**.

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Then M satisfies the assumptions in the discussion of the discrete series of disconnected groups, and M has **discrete series representations**.

The elements of M and A commute, so

$$MA = \{ma; m \in M, a \in A\}$$

is a subgroup of G, a **Levi factor**.

Restricted roots

For $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$, set

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}; \forall Y \in \mathfrak{a}, [Y, X] = \langle \alpha, Y \rangle X\}.$$

(We do not complexify now, and A is generally not a Cartan subgroup.)
The set of **restricted roots** of $(\mathfrak{a}, \mathfrak{g})$ is

$$\Sigma(\mathfrak{a}, \mathfrak{g}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq \{0\}\}.$$

A nilpotent subgroup

Fix a subset $\Sigma^+ \subset \Sigma(\mathfrak{a}, \mathfrak{g})$ of all restricted roots lying on one side of a hyperplane through 0, disjoint from $\Sigma(\mathfrak{a}, \mathfrak{g})$. The linear subspace

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \subset \mathfrak{g}$$

is a **nilpotent subalgebra**. This means that there is an $n \in \mathbb{N}$ such that for all $X_1, \ldots, X_n \in \mathfrak{n}$,

$$[X_1, [X_2, [X_3, [\cdots, X_n] \cdots] = 0.$$

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Let $N < G$ be the subgroup generated by $\exp(\mathfrak{n})$. It is **normalised** by MA.

Cuspidal parabolic subgroups

Definition

A **cuspidal parabolic subgroup** of G is a subgroup $P = MAN$ with M, A and N as on the previous slides.

There is a more general notion of a parabolic subgroup. Cuspidal means that M has discrete series representations.

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Properties:

- there are **finitely many** up to conjugacy, corresponding to the conjugacy classes of Cartan subgroups
- we have
$$G = KP,$$
so our definition of unitary induction applies
- the groups M, A and N have trivial intersections, and the multiplication map
$$M \times A \times N \to P$$
is a diffeomorphism.

III Examples of cuspidal parabolic subgroups

Example: compact Cartans

Suppose that $H = T < K$ is a **compact** Cartan subgroup. Then

- $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s} = \{0\}$
- $A = \exp(\mathfrak{a}) = \{e\}$
- $\mathfrak{m} = \mathfrak{g}$, the orthogonal complement to $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g}$
- $M_0 = G_0 = G$, the subgroup generated by $\exp(\mathfrak{m}) = \exp(\mathfrak{g})$
- $M = Z_K(\mathfrak{a})M_0 = G$.

And $N = \{e\}$ because $\mathfrak{a} = \{0\}$, so there are no restricted roots.

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And $N = \{e\}$ because $\mathfrak{a} = \{0\}$, so there are no restricted roots.

Now the factor $M = G$ indeed has discrete series representations.

So G is a cuspidal parabolic subgroup of itself if and only if G has discrete series representations.

Example: $\mathrm{SL}(2,\mathbb{R})$

For the compact Cartan subgroup $\mathrm{SO}(2) < \mathrm{SL}(2,\mathbb{R})$, the corresponding cuspidal parabolic subgroup is $\mathrm{SL}(2,\mathbb{R})$.

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Consider the noncompact (diagonal) Cartan subgroup

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} ; x \neq 0 \right\}.$$

Then $\mathfrak{h} = \mathbb{R}Y_2$, with

$$Y_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{s}.$$

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So

- $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s} = \mathfrak{h}$
- $A = \exp(\mathfrak{a}) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} ; r > 0 \right\}$
- $\mathfrak{m} = \{0\}$, the orthogonal complement to $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$
- $M_0 = G_0 = \{e\}$, the subgroup generated by $\exp(\mathfrak{m}) = \{e\}$
- $Z_K(\mathfrak{a}) = \{\pm I\}$
- $M = Z_K(\mathfrak{a}) M_0 = \{\pm I\}$ is **disconnected**.

Example: $SL(2, \mathbb{R})$, the group N

We had $\mathfrak{a} = \mathbb{R}Y_2$, with

$$Y_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{s}.$$

Let $\alpha \in \mathfrak{a}^*$ be defined by $\langle \alpha, Y_2 \rangle = 2$. Then $\Sigma(\mathfrak{a}, \mathfrak{g}) = \{\pm\alpha\}$, and

$$\mathfrak{g}_\alpha = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$\mathfrak{g}_{-\alpha} = \mathbb{R} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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Choose $\Sigma^+ = \{\alpha\}$. Then $\mathfrak{n} = \mathfrak{g}_\alpha$, and

$$N = \exp(\mathfrak{n}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{R} \right\}.$$

Example: $\mathrm{SL}(2,\mathbb{R})$, the group P

Now

$$P = MAN = \left\{ \pm I \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ; r > 0, x \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} y \neq 0, x \in \mathbb{R} \right\}.$$

Example: $\mathrm{SL}(3,\mathbb{R})$, the diagonal Cartan subgroup

For the the Cartan subgroup of $\mathrm{SL}(3,\mathbb{R})$ of diagonal matrices,

$$A = \exp(\mathfrak{h}) = \left\{ \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & (r_1 r_2)^{-1} \end{pmatrix} ; r_1, r_2 > 0 \right\}.$$

Then

$$M = Z_K(\mathfrak{a}) = \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 \end{pmatrix} ; \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \right\}.$$

The corresponding cuspidal parabolic subgroup consists of all upper-trangular matrices in $\mathrm{SL}(3,\mathbb{R})$, for a choice of Σ^+.

The arguments are analogous to $\mathrm{SL}(2,\mathbb{R})$, and extend to $\mathrm{SL}(n,\mathbb{R})$.

Example: $SL(3,\mathbb{R})$, another Cartan subgroup

Consider the Cartan subgroup H of matrices

$$\begin{pmatrix} x\cos t & -x\sin t & 0 \\ -x\sin t & x\cos t & 0 \\ 0 & 0 & x^{-2} \end{pmatrix}$$

with $t \in \mathbb{R}$ and $x \neq 0$. Then

$$\mathfrak{h} = \mathbb{R} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\in \mathfrak{k}} \oplus \mathbb{R} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\in \mathfrak{s}}.$$

Example: $\mathrm{SL}(3,\mathbb{R})$, the groups A and M_0

So

$$\mathfrak{a} = \mathbb{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$A = \exp(\mathfrak{a}) = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}; r > 0 \right\}$$

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And

$$Z_{\mathfrak{g}}(\mathfrak{a}) = \left\{\begin{pmatrix}X & 0\\ 0 & 0\end{pmatrix}; X \in \mathfrak{sl}(2,\mathbb{R})\right\} \oplus \mathfrak{a},$$

so

$$\mathfrak{m} = \left\{\begin{pmatrix}X & 0\\ 0 & 0\end{pmatrix}; X \in \mathfrak{sl}(2,\mathbb{R})\right\},$$

the orthogonal complement to $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a})$, and

$$M_0 = \mathrm{SL}(2,\mathbb{R}) \times \{1\},$$

the group generated by $\exp(\mathfrak{m})$.

Example: $\mathrm{SL}(3,\mathbb{R})$, the group M

Furthermore, $K = \mathrm{SO}(3)$, and

$$Z_K(\mathfrak{a}) = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} ; g \in \mathrm{O}(2) \right\}.$$

So

$$M = Z_K(\mathfrak{a})M_0 = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} ; g \in \mathrm{SL}(2,\mathbb{R})_{\pm} \right\}.$$

Here

$$\mathrm{SL}(2,\mathbb{R})_{\pm} = \{g \in \mathrm{GL}(2,\mathbb{R}); \det(g) \in \{\pm 1\}\}.$$

Again, M is disconnected.

Example: $SL(3,\mathbb{R})$, the group N

Now $\mathfrak{a} = \mathbb{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Let $\alpha \in \mathfrak{a}^*$ be such that its pairing with this matrix is 2. Then $\Sigma(\mathfrak{a}, \mathfrak{g}) = \{\pm\alpha\}$.

Example: $\mathrm{SL}(3,\mathbb{R})$, the group N

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And

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} ; x, y \in \mathbb{R} \right\},$$

and

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} ; x, y \in \mathbb{R} \right\}.$$

Choose $\Sigma^+ = \{\alpha\}$. Then $\mathfrak{n} = \mathfrak{g}_\alpha$, and

$$N = \exp(\mathfrak{n}) = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y \in \mathbb{R} \right\}.$$

Example: $\mathrm{SL}(3,\mathbb{R})$, the group P

The cuspidal parabolic subgroup corresponding to H is

$$P = MAN =$$

$$\left\{ \begin{pmatrix} & g & & 0 \\ & & & 0 \\ 0 & 0 & & \det(g)^{-1} \end{pmatrix} \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; \right.$$

$$\left. g \in \mathrm{SL}(2,\mathbb{R})_{\pm}, r > 0, ; x, y \in \mathbb{R} \right\}$$

$$= \left\{ \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array} \right) \right\} \subset \mathrm{SL}(3,\mathbb{R}).$$

Example: $\mathrm{SL}(n, \mathbb{C})$

Let $G = \mathrm{SL}(n, \mathbb{C})$. Like every complex group, it has only one conjugacy class of Cartan subgroups. We take the diagonal Cartan subgroup

$$H = \left\{ \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & z_n \end{pmatrix} ; z_j \in \mathbb{C}^\times, z_1 \cdots z_n = 1 \right\}.$$

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Then by similar computations as before,

$$A = \left\{ \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & r_n \end{pmatrix} ; r_j > 0, r_1 \cdots r_n = 1 \right\} \cong \mathbb{R}^{n-1};$$

$$M = \left\{ \begin{pmatrix} e^{i\alpha_1} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{i\alpha_n} \end{pmatrix} ; \alpha_j \in \mathbb{R}, \alpha_1 + \cdots + \alpha_n = 0 \right\} \cong \mathrm{U}(1)^{n-1}.$$

Example: $\mathrm{SL}(n, \mathbb{C})$ (cont'd)

For a choice of $\Sigma^+ \subset \Sigma(\mathfrak{a}, \mathfrak{g})$,

$$N = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} ; x_{jk} \in \mathbb{C} \right\}.$$

So P is the group of complex **upper triangular matrices** of size $n \times n$ with determinant 1:

$$P = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \right\} \subset \mathrm{SL}(n, \mathbb{C}).$$

Example: $\mathrm{SL}(n, \mathbb{C})$ (cont'd)

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In general, for a complex semisimple Lie group, there is only one conjugacy class of cuspidal parabolic subgroups. And the factor M is a torus.

IV Unitary induction from parabolic subgroups

Representations of P

Fix a cuspidal parabolic subgroup $P = MAN < G$. Let $\sigma \in \hat{M}$ and $\nu \in \hat{A}$. The map

$$\sigma \otimes \nu \otimes 1_N \colon P \to \mathrm{U}(\mathcal{H}_\sigma)$$

given by

$$(\sigma \otimes \nu \otimes 1_N)(man) = \sigma(m)\nu(a)$$

is a homomorphism because MA normalises N.

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The representations of G of the form

$$\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$$

will play an important role. The **principal series** of G is the set of these representations where P is minimal and $\sigma \in \hat{M}$ (then M is compact).

A ρ-element

Let $\Sigma^+ \subset \Sigma(\mathfrak{a}, \mathfrak{g})$ be the set of positive restricted roots used to define N. Write

$$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha.$$

Because $\ker(\exp_A) = \{0\}$, it defines a homomorphism

$$e^\rho \colon A \to (0, \infty).$$

The induced picture

Lemma

For $\sigma \in \hat{M}$ and $\nu \in \hat{A}$, the representation space $\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N)$ is the completion of the space of continuous functions $F \colon G \to \mathcal{H}_\sigma$ such that for all $g \in G$, $m \in M$, $a \in A$ and $n \in N$,

$$F(gman) = e^{-\rho}(a)\nu(a)^{-1}\sigma(m)^{-1}F(g)$$

in the inner product

$$(F_1, F_2) = \int_K (F_1(k), F_2(k))_{\mathcal{H}_\sigma}\, dk.$$

The induced picture, proof

We have $\Delta_G \equiv 1$ and $\Delta_P(man) = e^{2\rho}(a)$, so

$$\frac{\Delta_G^{1/2}}{\Delta_P^{1/2}}(man) = e^{-\rho}(a).$$

Hence the equivariance condition becomes

$$\begin{aligned} F(gman) &= \frac{\Delta_G^{1/2}}{\Delta_P^{1/2}}(man)(\sigma \otimes \nu \otimes 1_N)(man)^{-1}F(g) \\ &= e^{-\rho}(a)\nu(a)^{-1}\sigma(m)^{-1}F(g). \end{aligned}$$

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The equality

$$\int_G f(g)\, dg = \int_K \int_P f(kh) \frac{\Delta_P(man)}{\Delta_G(man)}\, dk\, dm\, da\, dn$$

used to prove that $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$ is unitary implies that the inner product on $\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N)$ equals the expression in the lemma.

The compact picture

Consider the space of continuous functions $F\colon K \to \mathcal{H}_\sigma$ such that for all $k \in K$ and $m \in M \cap K$,

$$F(km) = \sigma(m)^{-1} F(k).$$

Take its completion in the inner product

$$(F_1, F_2) = \int_K (F_1(k), F_2(k))_{\mathcal{H}_\sigma} \, dk.$$

This space is **independent of** ν.

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Take its completion in the inner product

$$(F_1, F_2) = \int_K (F_1(k), F_2(k))_{\mathcal{H}_\sigma}\, dk.$$

This space is **independent of** ν.

Define a representation of G in this space as follows. For $g \in G$ and $k' \in K$, write

$$g^{-1}k' = kman,$$

with $k \in K$, $m \in M$, $a \in A$ and $n \in N$. Then

$$(g \cdot F)(k') = e^{-\rho}(a)\nu(a)^{-1}\sigma(m)^{-1}F(k).$$

Lemma

This representation is equivalent to $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$.

The noncompact picture

Let

$$\bar{N} := \{(n^*)^{-1}; n \in N\}.$$

Then restriction to $\bar{N}$ is injective as a map on $\mathcal{H}_P^G(\sigma \otimes \nu \otimes 1_N)$, because $G \setminus \bar{N}MAN$ has measure zero. The image of this restriction map is the **noncompact picture** of $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$.

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The resulting representation space is relatively simple:

$$L^2(\bar{N}, \varphi d\bar{n})$$

for the weight function

$$\varphi(km\exp(X)n) = e^{2\,\mathrm{Re}(\langle \nu, X\rangle)}$$

for $k \in K$, $m \in M$, $X \in \mathfrak{a}$, $n \in N$ such that $km\exp(X)n \in \bar{N}$.

Example 1: the principal series of $SL(2, \mathbb{R})$

Consider the cuspidal parabolic subgroup $P = MAN < SL(2, \mathbb{R})$ of upper-triangular matrices, with

$$M = \{\pm I\}$$
$$A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}; r > 0 \right\} \cong \mathbb{R}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{R} \right\} \cong \mathbb{R}$$
$$\bar{N} = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}; x \in \mathbb{R} \right\} \cong \mathbb{R}.$$

Let σ_+ be the trivial representation of M, and σ_- the nontrivial irreducible one. Let $\nu \in \hat{A} \cong \mathbb{R}$.

Example 1a: the spherical principal series of $\mathrm{SL}(2,\mathbb{R})$

The representations $\mathrm{Ind}_P^G(\sigma_+ \otimes \nu \otimes 1_N)$, for $\nu \in \mathbb{R}$, are the **spherical principal series** of $\mathrm{SL}(2,\mathbb{R})$. Via the noncompact picture, the space $\mathcal{H}_P^G(\sigma_+ \otimes \nu \otimes 1_N)$ is isomorphic to $L^2(\mathbb{R})$. And for all

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

and $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\big(\mathrm{Ind}_P^G(\sigma_+ \otimes \nu \otimes 1_N)(g)f\big)(x) = |-bx+d|^{-1-i\nu} f\left(\frac{ax-c}{-bx+d}\right).$$

Example 1b: the non-spherical principal series of $\mathrm{SL}(2,\mathbb{R})$

The representations $\mathrm{Ind}_P^G(\sigma_- \otimes \nu \otimes 1_N)$, for $\nu \in \mathbb{R}$, are the **non-spherical principal series** of $\mathrm{SL}(2,\mathbb{R})$. As before,

$$\mathcal{H}_P^G(\sigma_- \otimes \nu \otimes 1_N) \cong L^2(\mathbb{R}).$$

And for all

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

and $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\left(\mathrm{Ind}_P^G(\sigma_- \otimes \nu \otimes 1_N)(g) f\right)(x) = \mathrm{sign}(-bx+d)|-bx+d|^{-1-i\nu} f\left(\frac{ax-c}{-bx+d}\right).$$

Example 2: the principal series of $\mathrm{SL}(2,\mathbb{C})$

Let $P = MAN < \mathrm{SL}(2,\mathbb{C})$ be the cuspidal parabolic of upper-triangular matrices. Then

$$M = \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} ; \alpha \in \mathbb{R} \right\} \cong \mathrm{U}(1)$$

$$A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} ; r > 0 \right\} \cong \mathbb{R}$$

$$N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} ; z \in \mathbb{C} \right\} \cong \mathbb{C}.$$

Example 2: the principal series of $\mathrm{SL}(2, \mathbb{C})$

Let $P = MAN < \mathrm{SL}(2, \mathbb{C})$ be the cuspidal parabolic of upper-triangular matrices. Then

$$M = \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} ; \alpha \in \mathbb{R} \right\} \cong \mathrm{U}(1)$$
$$A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} ; r > 0 \right\} \cong \mathbb{R}$$
$$N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} ; z \in \mathbb{C} \right\} \cong \mathbb{C}.$$

For $\sigma \in \hat{M} = \mathbb{Z}$ and $\nu \in \hat{A} = \mathbb{R}$, we have the principal series reresentation of $\mathrm{SL}(2, \mathbb{C})$ in $L^2(\mathbb{C})$ given by

$$\left(\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)(g) f\right)(z) = |-bz+d|^{-2-i\nu} \left(\frac{-bz+d}{|-bz+d|} \right)^{-\sigma} f\left(\frac{az-c}{-bz+d} \right).$$

Next lecture

In the third and last lecture, we use unitary induction from cuspidal parabolic subgroups to classify almost all tempered representations of G.

We will look at how to check which induced representations are **irreducible**, and when two of them are **equivalent**.