Tempered representations of real semisimple Lie groups
1. The discrete series

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The goal

The motivating goal of these lectures and Bram's is to explicitly describe

$$K_*(C_r^*(G)),$$

where

- G is a real semisimple Lie group
- $C_r^*(G)$ is its reduced group C^*-algebra
- K_* denotes K-theory.

Bram will discuss C^*-algebras and K-theory, I will discuss the representation theory needed to describe $C_r^*(G)$.

I Irreducible unitary representations of semisimple Lie groups

Real semisimple groups

In all of these lectures, G is a **connected, linear, real semisimple Lie group**:

- G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$
- G is closed under conjugate transpose
- G has finite centre.

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Examples:

- **compact:** $\mathrm{SO}(n)$, $\mathrm{SU}(n)$
- **complex:** $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$
- **noncompact, non-complex:** $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}(p, q)_0$, $\mathrm{SU}(p, q)$

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Non-examples:

- $\mathrm{O}(n)$
- the universal cover of $\mathrm{SL}(2, \mathbb{R})$
- the group of upper-triangular matrices in $\mathrm{SL}(n, \mathbb{R})$
- $\mathrm{GL}(n, \mathbb{C})$

Haar measure

There exists a measure dg on G such that for all $f \in C_c(G)$ and all $g' \in G$,

$$\int_G f(g'g)\,dg = \int_G f(g)\,dg.$$

Any two such measure are constant multiples of each other. We fix such a nonzero measure. This is a **left Haar measure** on G.

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Groups of the type we consider are **unimodular**: we also have

$$\int_G f(gg')\, dg = \int_G f(g)\, dg.$$

for all $f \in C_c(G)$ and all $g' \in G$.

Unitary representations

A **(continuous) unitary representation** of G is a group homomorphism

$$\pi\colon G \to \mathrm{U}(\mathcal{H}_\pi)$$

into the group of unitary operators on a Hilbert space $\mathcal{H}_\pi$, such that the map $G \times \mathcal{H}_\pi \to \mathcal{H}_\pi$ given by $(g, v) \mapsto \pi(g)v$ is continuous.

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Two unitary representations $\pi_j\colon G \to \mathrm{U}(\mathcal{H}_{\pi_j})$, for $j = 1, 2$ are **equivalent** if there is a unitary isomorphism $T\colon \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that for all $g \in G$,

$$T \circ \pi_1(g) = \pi_2(g) \circ T.$$

Examples of (non-)unitary representations

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The **trivial representation** $1_G \colon G \to \mathrm{GL}(1, \mathbb{C})$ given by $1_G(g) = 1$ is unitary.

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Suppose that G is **compact**, and that $\pi\colon G \to \mathrm{GL}(n, \mathbb{C})$ is a homomorphism. Define an inner product on $\mathbb{C}^n$ by

$$(v, w)_\pi := \int_G (\pi(g)v, \pi(g)w)_{\mathbb{C}^n}\, dg.$$

Then π is a unitary representation in $\mathbb{C}^n$ with the inner product $(-,-)_\pi$.

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Example

There is **no** inner product on $\mathbb{R}^n$ for which the standard representation of $\mathrm{SL}(n, \mathbb{R})$ in $\mathbb{R}^n$ is unitary.

The left and right regular representations

Consider the Hilbert space $L^2(G)$, defined w.r.t. the Haar measure dg. Define the **left and right regular representations** $L, R\colon G \to \mathrm{U}(L^2(G))$ by

$$(L(g)f)(g') = f(g^{-1}g')$$
$$(R(g)f)(g') = f(g'g),$$

for $g, g' \in G$ and $f \in L^2(G)$.

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These are unitary, where for R we use unimodularity of G.

We can view $L^2(G)$ as a unitary representation of $G \times G$ via the two commuting representations L and R.

Irreducible representations

Let $\pi\colon G \to \mathrm{U}(\mathcal{H}_\pi)$ be a unitary representation. Then π is **irreducible** if the only linear subspaces of $\mathcal{H}_\pi$ that are invariant under $\pi(G)$ are $\{0\}$ and $\mathcal{H}_\pi$.

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Example

Let $K < G$ be a nontrivial compact (strict) subgroup. Then the subspace

$$\{f \in L^2(G); \forall k \in K, R_k(f) = f\} \subset L^2(G)$$

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The set of equivalence classes of unitary irreducible representations of G is denoted by $\hat{G}$.

Decomposition into irreducible representations

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If G is **noncompact**, this is not true in general, e.g. for $L^2(G)$. We know from Fourier theory that $L^2(\mathbb{R})$ in some sense decomposes continuously into the unitary irreducible representations $x \mapsto e^{2\pi i \lambda x}$ of $\mathbb{R}$.

II Tempered representations

A Hilbert space

Consider the bundle of Hilbert spaces

$$\mathcal{E} := \coprod_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \to \hat{G}.$$

For a measure μ on $\hat{G}$, form the Hilbert space

$$\int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi)$$

of μ-measurable sections $\varphi \colon \hat{G} \to \mathcal{E}$ of this bundle (modulo equality almost everywhere), such that

$$\int_{\hat{G}} \|\varphi(\pi)\|^2_{\mathcal{H}_\pi \otimes \mathcal{H}_\pi^*} \, d\mu(\pi) < \infty.$$

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The representations $\pi \otimes \pi^*$ define a unitary representation of $G \times G$ in this space.

The Plancherel theorem

For every $f \in C_c^\infty(G)$, define its **Fourier transform** by

$$(\mathcal{F}(f))(\pi) := \pi(f) := \int_G f(g)\pi(g)\, dg \in \mathcal{B}(\mathcal{H}_\pi),$$

for $\pi \in \hat{G}$.

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Theorem

For all $f \in C_c^\infty(G)$ and $\pi \in \hat{G}$,

$$(\mathcal{F}(f))(\pi) \in \mathrm{HS}(\mathcal{H}_\pi) = \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*.$$

And there is a unique measure μ on $\hat{G}$ such that the Fourier transform extends to an equivalence of unitary representations of $G \times G$,

$$\mathcal{F} \colon L^2(G) \xrightarrow{\cong} \int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi).$$

The measure μ is the **Plancherel measure** of G.

Tempered representations

Definition

The set $\hat{G}_{\mathrm{temp}}$ of **tempered representations** of G is the support of the Plancherel measure on $\hat{G}$.

These lectures are about the classification of tempered representations of G.

Harish-Chandra classified almost all tempered representations and computed μ in the 1960s and 1970s. Knapp and Zuckerman classified all tempered representations in the 1980s.

Example 1: compact G

Suppose that G is compact. Then $\hat{G}_{\mathrm{temp}} = \hat{G}$ is countable and the Plancherel measure is the counting measure. So

$$L^2(G) \cong \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*.$$

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If $G = S^1$ (has infinite centre!), then $\hat{G} = \mathbb{Z}$ and this becomes the Fourier transform of functions in $L^2(S^1)$.

Example 2: $G = \mathbb{R}$

Suppose that $G = \mathbb{R}$ (infinite centre!).

Now $\hat{G}_{\text{temp}} = \hat{G} = \mathbb{R}$, where the representation corresponding to $\lambda \in \mathbb{R}$ is given by

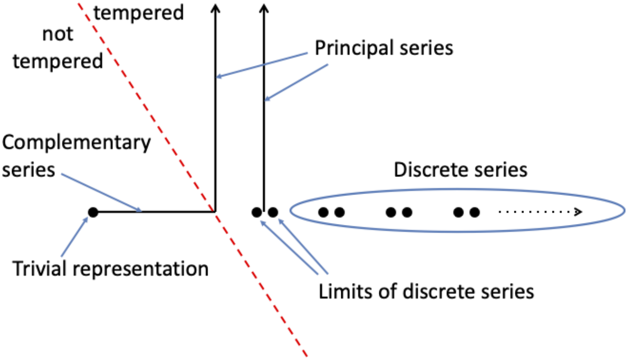
$$\pi_\lambda(x) = e^{2\pi i \lambda x} \in \mathrm{U}(\mathbb{C})$$

for $x \in \mathbb{R}$. The Plancherel measure is Lebesgue measure.

We then get the Fourier transform of functions in $L^2(\mathbb{R})$.

Example 3: $G = \mathrm{SL}(2, \mathbb{R})$

Suppose that $G = \mathrm{SL}(2, \mathbb{R})$, our main example. Then $\hat{G}$ looks like



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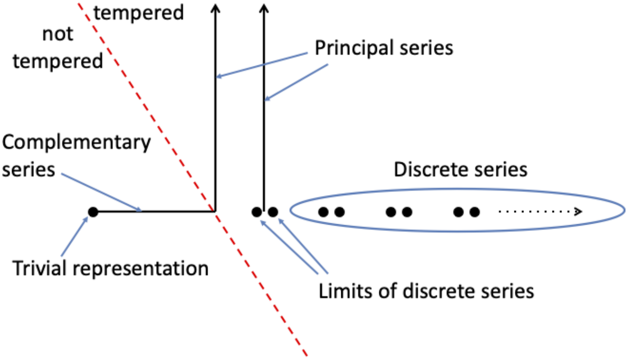

The Plancherel measure is

- $\frac{1}{8\pi}\nu \tanh(\pi\nu/2)\, d\nu$ on the left vertical half-line
- $\frac{1}{8\pi}\nu \coth(\pi\nu/2)\, d\nu$ on the right vertical half-line
- each point in the nth pair in the discrete series has measure $n/2\pi$.

The discrete series

Definition

A tempered representation π of G belongs to the **discrete series** $\hat{G}_{\mathrm{discr}}$ if the set $\{\pi\}$ has positive Plancherel measure.

So

$$\int_{\hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi) = \int_{\hat{G}_{\mathrm{temp}} \setminus \hat{G}_{\mathrm{discr}}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi) \oplus \bigoplus_{\pi \in \hat{G}_{\mathrm{ds}}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*.$$

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A tempered representation π of G belongs to the discrete series if and only if there are nonzero $v, w \in \mathcal{H}_\pi$ such that the **matrix coefficient**

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Our goal for this lecture is to classify the discrete series representations of G, if it has any. Even if G has no discrete series, then the discrete series of certain subgroups is used to describe $\hat{G}_{\mathrm{temp}}$.

Examples of discrete series

- If G is compact, then

$$\hat{G} = \hat{G}_{\text{temp}} = \hat{G}_{\text{discr}}.$$

- If $G = \mathbb{R}$, then $\hat{G}_{\text{discr}} = \emptyset$
- If $G = \text{SL}(2, \mathbb{R})$, then $\hat{G}_{\text{discr}}$ is the set of points away from the two half-lines
- If $G = \text{SL}(n, \mathbb{R})$ with $n \geq 3$, then $\hat{G}_{\text{discr}} = \emptyset$.

Why classify tempered representations?

- The representation $L^2(G)$ of $G \times G$ is important, e.g. for studying G-equivariant differental operators on spaces like G/H
- to describe the reduced C^*-algebra $C_r^*(G)$ of G
- to classify **admissible** representations.

Context: more general irreducible representations

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A possibly non-unitary irreducible representation is called **admissible** if in its restriction to a maximal compact subgroup $K < G$, every irreducible representation of K occurs finitely many times. Every unitary irreducible representation is admissible.

Admissible representations were classified by Langlands, in terms of tempered representations of subgroups.

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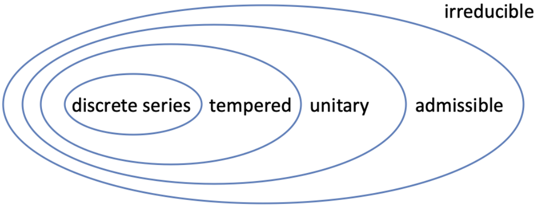
Non-admissible irreducible representations can behave very wildly, and are not studied as much.



III Cartan subgroups and roots

The Lie algebra of G

Suppose that $G < \mathrm{GL}(n, \mathbb{C})$. For $X \in M_n(\mathbb{C})$, we write

$$\exp(X) := \sum_{j=0}^{\infty} \frac{1}{j!} X^j.$$

The **Lie algebra** of G is

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For $X, Y \in \mathfrak{g}$ and $g \in G$,

$$\begin{aligned} \mathrm{Ad}(g)X &:= gXg^{-1} \in \mathfrak{g} \\ \mathrm{ad}(X)Y &:= [X, Y] := XY - YX \in \mathfrak{g}. \end{aligned}$$

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- A **subalgebra** of $\mathfrak{g}$ is a linear subspace closed under the bracket.
- A subalgebra is **commutative** if the bracket equals zero on it.
- G is **complex** if $\mathfrak{g}$ is a complex subspace of $M_n(\mathbb{C})$.

Examples of Lie algebras

- Because $\det(\exp(X)) = e^{\mathrm{tr}(X)}$,

$$\mathfrak{sl}(n, F) = \{X \in M_n(F); \mathrm{tr}(X) = 0\},$$

for $F = \mathbb{R}$ or $F = \mathbb{C}$.

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- Because $\frac{d}{dt}\Big|_{t=0} \exp(tX)\exp(tX)^* = X + X^*$,

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- So

$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}); X + X^* = 0, \mathrm{tr}(X) = 0\}.$$

Cartan subgroups and roots

Definition

A **Cartan subalgebra** of $\mathfrak{g}$ is a maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the operators $\{\mathrm{ad}(X), X \in \mathfrak{h}\}$ on $\mathfrak{g}$ diagonalise simultaneously.

A **Cartan subgroup** of G is the centraliser in G of a Cartan subalgebra.

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There are finitely many conjugacy classes of Cartan subgroups. All Cartan subgroups have the same dimension, this is the **rank** of G.

If G is compact or complex, then it has only one conjugacy class of Cartan subgroups. This is why representation theory is simpler for such groups.

Roots

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. For $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$, we write

$$\mathfrak{g}_\alpha^{\mathbb{C}} := \{X \in \mathfrak{g} \otimes \mathbb{C}; \forall Y \in \mathfrak{h}, [Y, X] = \langle \alpha, Y \rangle X\}.$$

Here the Lie brackets is extended complex-linearly to $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} + i\mathfrak{g}$.

Definition

A nonzero $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$ such that $\mathfrak{g}_\alpha^{\mathbb{C}} \neq \{0\}$ is a **root** of $(\mathfrak{h}, \mathfrak{g})$. The set of these roots is denoted by $R(\mathfrak{h}, \mathfrak{g})$.

The **root space decomposition** of $\mathfrak{g} \otimes \mathbb{C}$ w.r.t. $\mathfrak{h}$ is

$$\mathfrak{g} \otimes \mathbb{C} = (\mathfrak{h} \otimes \mathbb{C}) \oplus \bigoplus_{\alpha \in R(\mathfrak{h}, \mathfrak{g})} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

Example 1: a compact Cartan subgroup of $\mathrm{SL}(2,\mathbb{R})$

Suppose that $G = \mathrm{SL}(2,\mathbb{R})$. One Cartan subgroup of G is $T = \mathrm{SO}(2)$. Its Lie algebra is the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ spanned by

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Now $R(\mathfrak{t}, \mathfrak{g}) = \{\pm\alpha\}$, where

$$\langle \alpha, Y_1 \rangle = 2i.$$

We have

$$\mathfrak{g}^{\mathbb{C}}_{\alpha} = \mathbb{C} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad \mathfrak{g}^{\mathbb{C}}_{-\alpha} = \mathbb{C} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

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The corresponding root space decomposition is

$$\mathfrak{g}\otimes\mathbb{C} = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}Y_1 \oplus \mathbb{C}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Example 2: a noncompact Cartan subgroup of $\mathrm{SL}(2,\mathbb{R})$

Suppose again that $G = \mathrm{SL}(2,\mathbb{R})$. Another Cartan subgroup of G is

$$H := \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} ; x \neq 0 \right\}.$$

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Suppose again that $G = \mathrm{SL}(2,\mathbb{R})$. Another Cartan subgroup of G is

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Its Lie algebra is the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ spanned by

$$Y_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now $R(\mathfrak{h},\mathfrak{g}) = \{\pm\beta\}$, where

$$\langle \beta, Y_2 \rangle = 2.$$

We have

$$\mathfrak{g}^{\mathbb{C}}_{\beta} = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathfrak{g}^{\mathbb{C}}_{-\beta} = \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}Y_2 \oplus \mathfrak{g}^{\mathbb{C}}_{\beta} \oplus \mathfrak{g}^{\mathbb{C}}_{-\beta}$.

IV Classification of discrete series representations

Existence of the discrete series

Theorem (Harish-Chandra)

G has discrete series representations if and only if it has a compact Cartan subgroup.

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Examples:

- Every compact group has a compact Cartan subgroup: a maximal torus
- $\mathrm{SL}(2, \mathbb{R})$ has the compact Cartan subgroup $\mathrm{SO}(2)$
- $\mathrm{SU}(p, q)$ has the diagonal compact Cartan subgroup $\mathrm{U}(1)^{p+q-1}$
- if pq is even, then $\mathrm{SO}(p, q)_0$ has the compact Cartan subgroup $\mathrm{SO}(2)^{\lfloor p/2 \rfloor} \times \mathrm{SO}(2)^{\lfloor q/2 \rfloor}$
- if H is a Cartan subgroup of a complex Lie group, then $\mathfrak{h} \cong \mathbb{C}^n$ and $H \cong (\mathbb{C}^\times)^n$ for some n, so H is never compact.

Analytic integrality

Fix a compact Cartan subgroup $T < G$. An element $\xi \in i\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, i\mathbb{R})$ is **analytically integral** if

$$\langle \xi, \ker(\exp_T) \rangle \subset 2\pi i \mathbb{Z}.$$

The set of such elements is denoted by Λ.

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We have a bijection $\Lambda \cong \hat{T}$, where $\xi \in \Lambda$ corresponds to $e^\xi \colon T \to \mathrm{U}(1)$, given by

$$e^\xi(\exp(X)) = e^{\langle \xi, X \rangle},$$

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Example

Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, and $e^1, \ldots, e^n$ the dual basis vectors of $(\mathbb{R}^n)^*$. Let $T = \mathbb{R}^n/\mathbb{Z}^n$. Then $\ker(\exp_T) = \mathbb{Z}^n$. So

$$\Lambda = \mathrm{span}_{\mathbb{Z}}\{2\pi i e^1, \ldots, 2\pi i e^n\}.$$

Positive roots

Let $K := G \cap \mathrm{U}(n)$ be a maximal compact subgroup. The **Weyl group** of $R(\mathfrak{t}, \mathfrak{g})$ is

$$W(G, T) = N_G(T)/T = N_K(T)/T.$$

It acts on $i\mathfrak{t}^*$ via the adjoint action, and permutes the roots.

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We fix an inner product $(-, -)$ on $i\mathfrak{t}^*$ invariant under $W(G, T)$. Then any element $\xi \in i\mathfrak{t}^*$ such that $(\xi, \alpha) \neq 0$ for all $\alpha \in R(\mathfrak{t}, \mathfrak{g})$ determines a **positive root system** $R^+(\mathfrak{t}, \mathfrak{g}) \subset R(\mathfrak{t}, \mathfrak{g})$ by

$$\alpha \in R^+(\mathfrak{t}, \mathfrak{g}) \quad \Leftrightarrow (\xi, \alpha) > 0.$$

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$$\alpha \in R^+(\mathfrak{t}, \mathfrak{g}) \quad \Leftrightarrow (\xi, \alpha) > 0.$$

We set

$$\rho^\xi := \frac{1}{2} \sum_{\alpha \in R^+(\mathfrak{t}, \mathfrak{g})} \alpha$$

$$\rho_K^\xi := \frac{1}{2} \sum_{\alpha \in R^+(\mathfrak{t}, \mathfrak{g}), \mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}} \alpha.$$

Classification of the discrete series

Theorem (Harish-Chandra)

Suppose that G has a compact Cartan subgroup.

(a) *There is a discrete series representation π_λ for every $\lambda \in i\mathfrak{t}^*$ such that*
- $(\lambda, \alpha) \neq 0$ *for all* $\alpha \in R(\mathfrak{t}, \mathfrak{g})$, *and*
- $\lambda + \rho^\lambda$ *is analytically integral.*

(b) *All discrete series representations occur in this way.*

(c) *Two discrete series representations π_λ and $\pi_{\lambda'}$ are equivalent if and only if there is a $w \in W$ such that $\lambda' = w\lambda$.*

To make this theorem nontrivial, we need to specify a relation between λ and π_λ.

The parametrisation for compact groups: highest weights

Suppose for now that G is **compact**. Let $\pi \in \hat{G}$.

Analogously to the definition of roots, if $\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, i\mathbb{R})$, then the T-**weight space** of π for μ is

$$(\mathcal{H}_\pi)_\mu = \{v \in \mathcal{H}_\pi; \forall X \in \mathfrak{t}, \pi(X)v = \langle \mu, X \rangle v\}.$$

Here

$$\pi(X) = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX)).$$

And μ is a T-**weight** of π if $(\mathcal{H}_\pi)_\mu \neq 0$.

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Now the discrete series representation π_λ has **highest weight** $\lambda - \rho^\lambda$, i.e.

- $\dim(\mathcal{H}_{\pi_\lambda})_{\lambda-\rho^\lambda} = 1$
- all T-weights of π_λ are of the form

$$\lambda - \rho^\lambda - \sum_{\alpha \in R^+(\mathfrak{t},\mathfrak{g})} m_\alpha \alpha$$

for $m_\alpha \in \mathbb{Z}_{\geq 0}$.

The parametrisation in general: lowest K-types

For general $G < \mathrm{GL}(n, \mathbb{R})$ as before, consider the maximal compact subgroup $K = G \cap \mathrm{U}(n)$.

Then the **lowest K-type** of the discrete series representation π_λ has highest weight $\lambda + \rho^\lambda - 2\rho_K^\lambda$. I.e. every irreducible representation of K that occurs in $\pi_\lambda|_K$ has highest weight of the form

$$\lambda + \rho^\lambda - 2\rho_K^\lambda + \sum_{\alpha \in R^+(\mathfrak{t},\mathfrak{g})} m_\alpha \alpha,$$

for $m_\alpha \in \mathbb{Z}_{\geq 0}$.

Example: $\mathrm{SL}(2,\mathbb{R})$, ρ-elements

Consider the compact Cartan subgroup $T = \mathrm{SO}(2)$ of $\mathrm{SL}(2,\mathbb{R})$. Its Lie algebra $\mathfrak{t}$ is spanned by

$$Y_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\lambda := t\alpha/2 \in i\mathfrak{t}^*$. Then, by definition of α,

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We use the inner product

$$(x\alpha/2, y\alpha/2) = xy$$

on $i\mathfrak{t}^*$. Then $(\lambda, \alpha) = 2t \neq 0$ if and only if $t \neq 0$. Now

$$\rho^\lambda = \begin{cases} \alpha/2 & \text{if } t > 0 \\ -\alpha/2 & \text{if } t < 0. \end{cases}$$

Example: $SL(2, \mathbb{R})$, classification of the discrete series

We have

$$\exp(sY_1) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

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Let $\lambda = t\alpha/2$ with $t > 0$. Then $\lambda + \rho^\lambda = (t+1)\alpha/2$. This element is analytically integral if and only if

$$2\pi i(t+1) = \langle \lambda + \rho^\lambda, 2\pi Y_1 \rangle \in 2\pi i\mathbb{Z},$$

so $t = n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We have the corresponding **holomorphic discrete series** representation D_n^+ of $SL(2, \mathbb{R})$.

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If $\lambda = t\alpha/2$ with $t < 0$. Then $\lambda + \rho^\lambda = (t-1)\alpha/2$. This element is analytically integral if and only if $-t = n \in \mathbb{N}$. We have the corresponding **antiholomorphic discrete series** representation D_n^-.

Example: $\mathrm{SL}(2, \mathbb{R})$, Weyl group

Now $K = T$, so

$$W = N_K(T)/T = \{e\}.$$

So none of the representations D_n^+ and D_n^- are equivalent.

Example: $SL(2, \mathbb{R})$, realisation of the discrete series

Let $n \in \mathbb{N}$. Consider the inner product

$$(f_1, f_2)_n := \int_{\mathbb{H}} f_1(z)\overline{f_2(z)}\,\mathrm{Re}(z)^{n-1}\,dz$$

for functions on the upper half plane $\mathbb{H} = \{\mathrm{Re}(z) > 0\}$. Consider the Hilbert space

$$\mathcal{H}_n^+ := \{f\colon \mathbb{H} \to \mathbb{C}; \text{holomorphic}, (f, f)_n < \infty\}.$$

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Then D_n^+ is the representation of $\mathrm{SL}(2,\mathbb{R})$ in $\mathcal{H}_n$ given by

$$(D_n^+(g)f)(z) = (-bz+d)^{-n-1} f\left(\frac{az-c}{-bz+d}\right).$$

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Let $\mathcal{H}_n^- = \overline{\mathcal{H}_n^+}$ be the Hilbert space defined as $\mathcal{H}_n^+$ but with antiholomorphic functions. The representation D_n^- is the representation in $\mathcal{H}_n^-$ given by the same expression as D_n^+.

Summary

Important points in this lecture:

- The **tempered representations** of G are the unitary irreducible representations that occur in the Plancherel decomposition of $L^2(G)$.
- If G has a compact Cartan subgroup, then its has a special type of tempered representations: the **discrete series**. These give direct summands of $L^2(G)$.
- If G has discrete series representations, then these form a countable set, and we can **classify** them explicitly.