EW fluctuations for the AKPZ equation in the weak coupling regime

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PDE and Randomness

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The KPZ equation

$$\partial_t h = \nu \Delta h + \lambda \langle \nabla h, Q \nabla h \rangle + \sqrt{D} \xi \qquad \textbf{(KPZ)}$$

- $h = h(t, x)$, $t \geq 0$ and $x \in \mathbb{R}^d$ (or $\mathbb{T}_L^d$),
- ξ is a space-time white noise,
- ν, $D > 0$, $\lambda \geq 0$ and Q is a $d \times d$ symmetric matrix.

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$\rightsquigarrow$ **K**ardar-**P**arisi-**Z**hang surfaces

ν smoothing mechanism

Q slope-dependence

λ strength of non-linearity (= coupling constant)

D noise strength

The KPZ equation

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The KPZ equation

$\rightsquigarrow \nu = \frac{1}{2}$ and $D = 1$

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$\longrightarrow$ KPZ is ill-posed in *any* dimension!

$$h \sim -\frac{d}{2} + 1 \qquad \Longrightarrow \qquad \langle \nabla h, Q\nabla h\rangle = \textbf{??}$$

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$\longrightarrow$ In $d = 1$ a variety of approaches:

- Cole-Hopf Transform Bertini-Cancrini ('95), Bertini-Giacomin ('96)
- Martingale Problem Gonçalves-Jara ('14), Gubinelli-Jara ('13), Gubinelli-Perkowski ('18)
- Pathwise Approach Rough paths, Regularity Structures (Hairer '13-'14), Paracontrolled Calculus (Gubinelli-Perkowski '17)

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$\longrightarrow$ These break down for $d > 1$!

Goal

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\langle \nabla h, Q\nabla h\rangle + \xi \qquad \textbf{(KPZ)}$$

Questions

- What is the large-scale behaviour of KPZ?
- What is the role of the nonlinearity? Is it *relevant* ?

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Heuristics

$\longrightarrow$ *Diffusive scaling* : $h \quad \mapsto \quad h^\varepsilon(t,x) = \varepsilon^{1-\frac{d}{2}} h(t/\varepsilon^2, x/\varepsilon)$

$$\partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \underbrace{\lambda\varepsilon^{\frac{d-2}{2}}}_{=\lambda_\varepsilon}\langle \nabla h^\varepsilon, Q\nabla h^\varepsilon\rangle + \xi$$

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Prediction:

$d = 1$ $\lambda_\varepsilon \uparrow \infty$ *relevant*

$d \geq 3$ $\lambda_\varepsilon \downarrow 0$ *irrelevant*

$d = 2$ $\lambda_\varepsilon = \lambda$ *marginal*

How to measure **relevance**

(KPZ) $\partial_t h = \frac{1}{2}\Delta h + \lambda\langle\nabla h, Q\nabla h\rangle + \xi$, $\partial_t \mathfrak{h} = \frac{\nu}{2}\Delta\mathfrak{h} + \sqrt{\nu}\xi$ **(SHE)**

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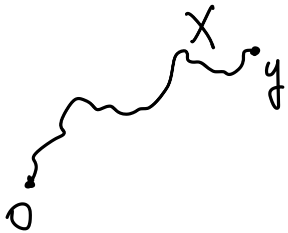
The Bulk Diffusion coefficient

$$\mathrm{D}_{\mathrm{bulk}}(t) \stackrel{\text{def}}{=} \frac{1}{2t}\int |x|^2 \mathbb{E}[(-\Delta)^{\frac{1}{2}} h(t,x)(-\Delta)^{\frac{1}{2}} h(0,0)]\mathrm{d}x$$



$$\int |y|^2 P_t(y)\,dy = t \qquad D(t) = \frac{1}{t}\int |y|^2 P_t(y)$$

$$S(t,x) = \mathbb{E}\left[(-\Delta)^{1/2} h(t,x)\,(-\Delta)^{1/2} h(0,0)\right]$$

$$S(0,x) = \delta_0(x)$$

$$D^{SHE}(t) = \nu$$

The subcritical case: $d = 1$

$$\partial_t h = \frac{1}{2}\Delta h + \lambda(\partial_x h)^2 + \xi \qquad \textbf{(KPZ)}$$

KPZ is *relevant*

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$\longrightarrow$ Super-Diffusivity: $\frac{1}{C} t^{\frac{1}{3}} \leq \mathrm{D}_{\mathrm{bulk}}(t) \leq t^{\frac{1}{3}} C$ $\rightsquigarrow$ Balász-Quastel-Seppäläinen '11

$\longrightarrow$ Large-scale distributions: explicit and NOT Gaussian $\rightsquigarrow$ Amir-Corwin-Quastel '11, Sasamoto-Spohn '11,...

$\longrightarrow$ Limit process: KPZ Fixed Point $\rightsquigarrow$ Quastel-Sarkar '20, Viràg '20,...

The supercritical case: $d \geq 3$

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\langle\nabla h, Q\nabla h\rangle + \xi \qquad \textbf{(KPZ)}$$

KPZ is *irrelevant* (at least for λ small enough)

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$|\nabla h|^2$

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$Q = \mathrm{Id} \rightsquigarrow$ Magnen-Unterberger '18, Dunlap-Gu-Ryzhik-Zeitouni '18, Comets-Cosco-Mukherjee '19, Lygkonis-Zygouras '20, Cosco-Nakajima-Nakashima '20

Theorem

For λ small enough $h^\varepsilon(t,x) \stackrel{\text{def}}{=} \varepsilon^{1-\frac{d}{2}} h^1(t/\varepsilon^2, x/\varepsilon)$, solution of

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \lambda\varepsilon^{\frac{d-2}{2}}|\nabla h^\varepsilon|^2 + \xi * \phi^{1/\varepsilon}$$

satisfies

$$h^\varepsilon - C_\varepsilon t \xrightarrow{\varepsilon\downarrow 0} \mathfrak{h}_\lambda \quad \text{where} \quad \partial_t \mathfrak{h}_\lambda = \tfrac{1}{2}\Delta\mathfrak{h}_\lambda + \sqrt{D_{\mathrm{eff}}}\xi$$

The critical dimension $d = 2$: Wolf's conjecture

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KPZ is *marginal*

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Importance of Q (Wolf '91)

- $\det Q > 0$ Isotropic KPZ Class $\rightsquigarrow$ *relevant*
- $\det Q \leq 0$ Anisotropic KPZ Class $\rightsquigarrow$ *irrelevant*

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IKPZ directed polymers, deposition models,...

$\rightsquigarrow$ Caravenna-Sun-Zygouras '20, Chatterjee-Dunlap '20, Gu '20 ...

AKPZ intelaced particle systems, Gates Wescott model, domino shuffling algorithm, q-Whittaker processes,...

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$\longrightarrow$ NO result on space-time correlations

The AKPZ equation

Focus $Q = \mathrm{diag}(1, -1)$

$$\partial_t h = \tfrac{1}{2}\Delta h + \lambda\Big[(\partial_1 h)^2 - (\partial_2 h)^2\Big] + \xi \qquad \textbf{(AKPZ)}$$

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Proposition (C.-Erhard-Schönbauer '19)

For *any choice* of $\lambda > 0$, (**AKPZ**) admits a unique solution h such that

- h exists globally in time,
- h is a strong Markov process,
- the Gaussian Free Field η is its invariant measure.

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Remark

- η GFF $\longrightarrow \eta(\phi) \overset{\text{law}}{=} N(0, \langle \phi, (-\Delta)^{-1}\phi\rangle)$.
- Invariance:

$$\langle \Pi_1 \Big[(\partial_1 \Pi_1 \eta)^2 - (\partial_2 \Pi_1 \eta)^2\Big], \eta\rangle_{H^1(\mathbb{T}^2)} = 0\,.$$

Super-diffusive large scale behaviour

The **bulk-diffusion coefficient** $D(t)$ is

$$\mathrm{D}_{\mathrm{bulk}}(t) \stackrel{\mathrm{def}}{=} \frac{1}{2t}\int |x|^2 \mathbb{E}\Big[(-\Delta)^{\frac{1}{2}} h(t,x)(-\Delta)^{\frac{1}{2}} h(0,0)\Big] \mathrm{d}x$$

Theorem (C.-Erhard-Toninelli '20-'21)

There exists a constant $C > 0$ such that for all $\lambda > 0$ and $\delta > 0$, in a weak Tauberian sense,

$$\frac{1}{C}(\log\log t)^{-5-\delta}\sqrt{\log t} \;\le\; \mathrm{D}_{\mathrm{bulk}}(t) \;\le\; \sqrt{\log t}\, C\,(\log\log t)^{5+\delta}$$

for t large enough.

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Remarks

- $\longrightarrow$ AKPZ is not diffusive $\rightsquigarrow$ Borodin-Corwin-Ferrari '18
- $\longrightarrow$ Temporal scaling

$$h^{\varepsilon}(t,x) = h\left(\frac{t}{\varepsilon^2\sqrt{\log\varepsilon^{-1}}}, \frac{x}{\varepsilon}\right)$$

The Weak Coupling scaling

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\mathcal{N}^1[h] + \xi \qquad \textbf{(AKPZ)}$$

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$\rightsquigarrow$ under diffusive scaling $h^\varepsilon(t,x) = h(t/\varepsilon^2, x/\varepsilon)$

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For all $T > 0$ and any $\lambda > 0$

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Remark $|\nabla h|^2$

$\longrightarrow$ For $Q = \mathrm{Id}$ same scaling considered in Chatterjee-Dunlap '19, Caravenna-Sun-Zygouras '19, Gu '19

$\longrightarrow$ There is no phase transition in λ in AKPZ ($\neq$ IKPZ !).

What is the limit?

Recall

$$\mathrm{D}^{\varepsilon}_{\mathrm{bulk}}(t) \stackrel{\text{def}}{=} \frac{1}{2t}\int |x|^2 \mathbb{E}\Big[(-\Delta)^{\frac{1}{2}} h^{\varepsilon}(t,x)(-\Delta)^{\frac{1}{2}} h^{\varepsilon}(0,0)\Big]\mathrm{d}x$$

Theorem (C.-Erhard-Toninelli '21)

For all $t > 0$ and any $\lambda > 0$

$$\lim_{\varepsilon\to 0} \mathrm{D}^{\varepsilon}_{\mathrm{bulk}}(t) = \nu_{\mathrm{eff}}$$

where ν_{eff} is given by

$$\nu_{\mathrm{eff}} \stackrel{\text{def}}{=} \sqrt{\frac{2\lambda^2}{\pi} + 1}$$

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Remark

For all $\lambda > 0$,

$\longrightarrow$ it is well-defined $\Longrightarrow$ no phase transition

$\longrightarrow$ $\nu_{\mathrm{eff}} > 1$

Main Result

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \frac{\lambda}{\sqrt{\log \varepsilon^{-1}}}\mathcal{N}^{\varepsilon^{-1}}[h^\varepsilon] + \xi \qquad \textbf{(AKPZ)}$$

Theorem (C.-Erhard-Toninelli '21)

For any $T > 0$ and $\lambda > 0$,

$$h^\varepsilon \overset{\text{law}}{\Longrightarrow} \mathfrak{h}_\lambda\,, \qquad \text{as } \varepsilon \to 0$$

in $C([0,T], \mathcal{S}'(\mathbb{T}^2))$, where $\mathfrak{h}_\lambda$ is the (stationary) solution to

$$\partial_t \mathfrak{h}_\lambda = \frac{\nu_{\text{eff}}}{2}\Delta \mathfrak{h}_\lambda + \sqrt{\nu_{\text{eff}}}\xi$$

and $\nu_{\text{eff}} = \sqrt{\frac{2\lambda^2}{\pi} + 1}$.

SHE

$\longrightarrow$ By [C.-Erhard-Schönbauer '19],

$$\{h^\varepsilon\}_\varepsilon \text{ is tight} \quad \Longrightarrow \quad h^\varepsilon \longrightarrow \mathfrak{h} \quad \text{in } C([0,T],\mathcal{S}'(\mathbb{T}^2)) \text{ along subsequences}$$

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Martingale characterisation of SHE e.g. Mourrat-Weber '16

If for all $\phi \in \mathcal{C}^\infty(\mathbb{T}^2)$

$F(\eta) := \eta(\phi)$

$F(h_t) - F(\eta) - \int_0^t \mathcal{L}_0 F(h_s)\, ds$

$$M_t^\phi \overset{\text{def}}{=} \mathfrak{h}_t(\phi) - \eta(\phi) - \int_0^t \mathcal{L}_0^{\nu_{\text{eff}}} \mathfrak{h}_s(\phi)\mathrm{d}s \quad , \quad \Gamma_\phi(t) \overset{\text{def}}{=} (M_t^\phi)^2 - t\,\nu_{\text{eff}}\,\|\phi\|^2_{L^2(\mathbb{T}^2)}$$

are local martingales, then

$$\mathfrak{h} \overset{\text{law}}{=} \mathfrak{h}_\lambda$$

where $\mathcal{L}_0^{\nu_{\text{eff}}}\mathfrak{h}_s(\phi) \overset{\text{def}}{=} \frac{\nu_{\text{eff}}}{2}\mathcal{L}_0 \mathfrak{h}_s(\phi) = \frac{\nu_{\text{eff}}}{2}\mathfrak{h}_s(\Delta\phi)$

Observables

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \frac{\lambda}{\sqrt{\log \varepsilon^{-1}}}\mathcal{N}^{\varepsilon^{-1}}[h^\varepsilon] + \xi\,, \qquad h_0^\varepsilon \stackrel{\text{def}}{=} \eta \qquad \textbf{(AKPZ)}$$

$F(\eta) \stackrel{\text{def}}{=} f(\eta(\phi_1), \dots, \eta(\phi_n))$ cylinder,

$$F(h_t^\varepsilon) - F(\eta) - \int_0^t \mathcal{L}^\varepsilon F(h_s^\varepsilon)\mathrm{d}s = \mathcal{M}_t(F)$$

where $\mathcal{L}^\varepsilon$ is the **generator** of h^ε.

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$\longrightarrow M_t^\phi \stackrel{\text{def}}{=} \mathfrak{h}_t(\phi) - \mathfrak{h}_0(\phi) - \int_0^t \mathcal{L}_0^{\nu_{\text{eff}}} \mathfrak{h}_s(\phi)\mathrm{d}s$

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$\nu_{eff} > 1$

Naive attempt: *Choose $F(\eta) \stackrel{\text{def}}{=} \eta(\phi)$.*

$$h_t^\varepsilon(\phi) - \eta(\phi) - \frac{1}{2}\int_0^t h_s^\varepsilon(\Delta\phi)\mathrm{d}s - \frac{\lambda}{\sqrt{\log \varepsilon^{-1}}}\int_0^t \mathcal{N}^{\varepsilon^{-1}}[h^\varepsilon](\phi)\mathrm{d}s = \int_0^t \xi_s(\phi)\mathrm{d}s$$

$$\downarrow$$

$$h_t(\phi) - \eta(\phi) - \frac{1}{2}\int_0^t h_s(\Delta\phi)\,ds \qquad = \int_0^t \xi_s(\phi)\,ds$$

The generator of h^ε

Wiener chaos analysis

- $\rightsquigarrow$ the invariant measure is η, a Gaussian FF,
- $\rightsquigarrow$ $L^2(\eta) = \bigoplus_n \mathscr{H}_n$ s.t. $\mathscr{H}_n \perp \mathscr{H}_m$ for $m \neq n$,
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The generator of h^ε

Wiener chaos analysis

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Generator

- $\rightsquigarrow$ h^ε is Markov and its generator $\mathcal{L}^\varepsilon$ is such that

$$\mathcal{L}^\varepsilon = \underbrace{\mathcal{L}_0}_{\text{symmetric}} + \underbrace{\mathcal{A}_+^\varepsilon + \mathcal{A}_-^\varepsilon}_{\text{antisymmetric}}$$

generator of SHE

- $\rightsquigarrow$ $\mathcal{L}_0$, $\mathcal{A}_+^\varepsilon$, $\mathcal{A}_-^\varepsilon$ satisfy for all n

$$\mathcal{L}_0 : \mathscr{H}_n \to \mathscr{H}_n \qquad \mathcal{A}_+^\varepsilon : \mathscr{H}_n \to \mathscr{H}_{n+1} \qquad \mathcal{A}_-^\varepsilon : \mathscr{H}_n \to \mathscr{H}_{n-1}$$

The generator equation

Recall

$$F(h_t^\varepsilon) - F(\eta) - \int_0^t \mathcal{L}^\varepsilon F(h_s^\varepsilon)\mathrm{d}s = \mathcal{M}_t(F) \quad , \quad \mathfrak{h}_t(\phi) - \eta(\phi) - \int_0^t \mathcal{L}_0^{\nu_{\mathrm{eff}}} \mathfrak{h}_s(\phi)\mathrm{d}s = M_t^\phi$$

and

$$\mathbb{E}[\langle \mathcal{M}_\cdot(F)\rangle_t] = 2t\|(-\mathcal{L}_0)^{1/2}F\|$$

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$$F^\varepsilon(h_t^\varepsilon) - F^\varepsilon(\eta) - \int_0^t \mathcal{L}_0^{\nu_{\mathrm{eff}}} h_s^\varepsilon(\phi)\mathrm{d}s = \mathcal{M}_t(F)$$

Problems

- *Does such an F^ε exist?*
- $F^\varepsilon \longrightarrow$ *??*
- $\mathcal{M}_t(F^\varepsilon) \longrightarrow$ *??*

The truncated generator equation

Idea: Consider $P_n : L^2(\eta) \to \bigoplus_{j\leq n} \mathscr{H}_j$ and the **truncated generator**

$$\mathcal{L}_n^\varepsilon \stackrel{\text{def}}{=} \mathrm{P}_n \mathcal{L}^\varepsilon \mathrm{P}_n$$

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Advantages

$\longrightarrow$ $F^n \in \bigoplus_{j\leq n} \mathscr{H}_j \implies$ finitely many equations!

$\longrightarrow$ An (almost) explicit description of F^n as

$$\begin{aligned} -\mathcal{L}_0 F_n^{\varepsilon,n} - \mathcal{A}_+^N F_{n-1}^{\varepsilon,n} &= 0 \\ -\mathcal{L}_0 F_{n-1}^{\varepsilon,n} - \mathcal{A}_-^N F_n^{\varepsilon,n} - \mathcal{A}_+^N F_{n-2}^{\varepsilon,n} &= 0 \\ &\cdots \\ -\mathcal{L}_0 F_1^{\varepsilon,n} - \mathcal{A}_-^N F_2^{\varepsilon,n} &= \mathcal{L}_0^{\nu_{\text{eff}}} \eta(\phi) \end{aligned}$$

The truncated generator equation

$F^n = (F^n_j)_j$ solves $\mathcal{L}^\varepsilon F^n = \frac{\nu_{\text{eff}}}{2}\eta(\Delta\phi)$ iff

$$
\begin{aligned}
F^{\varepsilon,n}_n &= (-\mathcal{L}_0 + \mathcal{H}^\varepsilon_2)^{-1}\mathcal{A}^\varepsilon_+ F^{\varepsilon,n}_{n-1},\\
F^{\varepsilon,n}_{n-1} &= (-\mathcal{L}_0 + \mathcal{H}^\varepsilon_3)^{-1}\mathcal{A}^\varepsilon_+ F^{\varepsilon,n}_{n-2}\\
&\dots\\
F^{\varepsilon,n}_2 &= (-\mathcal{L}_0 + \mathcal{H}^\varepsilon_n)^{-1}\mathcal{A}^\varepsilon_+ F^{\varepsilon,n}_1\\
F^{\varepsilon,n}_1 &= (-\mathcal{L}_0 + \mathcal{H}^\varepsilon_{n+1})^{-1}\mathcal{L}^{\nu_{\text{eff}}}_0\eta(\phi),
\end{aligned}
$$

$$\mathcal{H}^\varepsilon_2 \stackrel{\text{def}}{=} 0$$

$$\mathcal{H}^\varepsilon_j \stackrel{\text{def}}{=} -\mathcal{A}^\varepsilon_-(-\mathcal{L}_0 + \mathcal{H}^\varepsilon_{j-1})^{-1}\mathcal{A}^\varepsilon_+$$

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Proposition (C.-Erhard-Toninelli ’21)

For each $j \geq 2$, there exists a function G^ε_j such that

$$\mathcal{H}^\varepsilon_j \quad \approx \quad -\mathcal{L}_0 \; G^\varepsilon_j(-\mathcal{L}_0)$$

- for all x fixed, $\lim_\varepsilon G^\varepsilon_j(x) = c_j$ (but the convergence is NOT uniform!)
- $\lim_{j\to\infty} c_j = \nu_{\text{eff}} - 1$

The behaviour of the solution to the TGE

Theorem (C.-Erhard-Toninelli '21)

1	$F^{\varepsilon,n}$	$\xrightarrow{\varepsilon}$	$\frac{\nu_{\text{eff}}}{1+c_n}\eta(\phi)$	$\xrightarrow{n}$	$\eta(\phi)$
2	$\|(-\mathcal{L}_0)^{1/2}F_n^{\varepsilon,n}\|$	$\xrightarrow{\varepsilon}$	C_n	$\xrightarrow{n}$	0
3	$\|(-\mathcal{L}_0)^{1/2}F^{\varepsilon,n}\|$	$\xrightarrow{\varepsilon}$	$\sum_{j=1}^{n} C_j$	$\xrightarrow{n}$	ν_{eff}

$$\rightsquigarrow \quad F^{\varepsilon,n}(h_t^\varepsilon) - F^{\varepsilon,n}(\eta) - \int_0^t \mathcal{L}_n^\varepsilon F^{\varepsilon,n}(h_s^\varepsilon)\mathrm{d}s - \int_0^t \mathcal{A}_+^N F_n^{\varepsilon,n}(h_s^\varepsilon)\mathrm{d}s = \mathcal{M}_t(F^{\varepsilon,n}),$$

$$h_t(\phi) - \eta(\phi) - \int_0^t \mathcal{L}_0^{\text{eff}} h_s(\phi)ds \qquad 0$$

- $\mathbb{E}[\langle \mathcal{M}_\cdot(F^{\varepsilon,n})\rangle_t] = t\|(-\mathcal{L}_0)^{1/2}F^{\varepsilon,n}\|$

Open questions and prospects

- $\longrightarrow$ EW fluctuations in *weak coupling* for *general* Q, $\det Q \leq 0$.
- $\longrightarrow$ EW fluctuations in *weak coupling* for *critical/supercritical* SPDEs.
- $\longrightarrow$ Gaussianity in *strong coupling*.

Thank you for your attention!