Diffusion in the curl of the 2-dimensional Gaussian Free Field

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joint work with G. Cannizzaro (Warwick) and L. Haunschmid (TU Wien)

Workshop “PDEs and Randomness”

Bath (kind of), September 2021

The equation: informal version

Our main character: **a Brownian diffusion in random environment**

$$dX_t = \omega(X_t)dt + dB_t, \qquad X_0 = 0$$

where:

- $X_t = (X_t^{(1)}, X_t^{(2)}) \in \mathbb{R}^2$: the tracer particle position
- B_t: standard two-dimensional Brownian motion
- $\omega = (\omega_1, \omega_2)$: random vector field (drift), law $\mathbb{P}$

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The driving field is given by

$$\omega_1(x) = \partial_{x_2}\tilde{h}(x), \quad \omega_2(x) = -\partial_{x_1}\tilde{h}(x)$$

where $\tilde{h}$ is Gaussian Free Field (GFF) on the plane: for zero-mean test function φ,

$$\tilde{h}(\varphi) \stackrel{d}{=} \mathcal{N}(0, \langle\varphi, (-\Delta)^{-1}\varphi\rangle).$$

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Note:

$$\omega = \nabla \times (0, 0, \tilde{h}).$$

The equation: regularized version

$$dX_t = \omega(X_t)dt + dB_t, \qquad \omega = \nabla \times (0,0,\tilde{h}), \qquad \tilde{h} = GFF$$

Problem

$\tilde{h}$ is a distribution, the drift field is too irregular [space regularity -1^-, not enough]

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Solution: regularize the GFF

Regularization

- take $U : \mathbb{R}^2 \mapsto \mathbb{R}$ with $\int U(x)dx = 1$, approximate delta function (smooth, radial)
- define $h(x) = (U * \tilde{h})(x)$ (convolution)
- now $\omega = \nabla \times (0,0,h)$ is well-defined and smooth.

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Notation

$\mathbb{P}$: law of the environment ω.

Goal and conjectures

$$dX_t = \omega(X_t)dt + dB_t, \qquad \omega = \nabla \times (0,0,h), \qquad h = (GFF)_{reg}$$

Questions

- What is the large-scale behavior of X?
- Diffusion/superdiffusion?

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Conjecture: Logarithmic super-diffusion

$$\mathbf{E}(|X(t)|^2) \overset{t\to\infty}{\sim} t \times \sqrt{\log t}$$

[B. Tóth, B. Valkó, 2012]

E: joint expectation w.r.t. environment and Brownian noise (“annealed”)

Known results

B. Tóth and B. Valkó (2012) proved:

$$\log\log t \lesssim \frac{1}{t}\mathbf{E}(|X(t)|^2) \lesssim \log t, \qquad t \to \infty$$

in Tauberian sense, i.e.

$$\lambda^{-2}\log|\log\lambda| \lesssim D(\lambda) \lesssim \lambda^{-2}|\log\lambda|$$

as $\lambda \to 0^+$, where

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NB: in Laplace transform, the conjecture is

$$D(\lambda) \overset{\lambda\to 0^+}{\approx} \lambda^{-2}\sqrt{|\log\lambda|}.$$

Our result

Theorem (G. Cannizzaro, L. Haunschmid, F.T. '21, arXiv)

For every $\varepsilon > 0$, as $\lambda \to 0^+$,

$$c_-(\varepsilon)(\log|\log\lambda|)^{-3/2-\varepsilon} \le \lambda^2 \frac{D(\lambda)}{\sqrt{|\log\lambda|}} \le c_+(\varepsilon)(\log|\log\lambda|)^{+3/2+\varepsilon}.$$

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In "real time" one can deduce:

- upper bound:

$$\mathbf{E}(|X(t)|^2) \le C(\varepsilon) t\sqrt{\log t}(\log\log t)^{3/2+\varepsilon}$$

  using argument by Tarrès, Tóth, Valkó 2012

- lower bound:

$$\limsup_{t\to\infty} \frac{\mathbf{E}(|X(t)|^2)}{t\sqrt{\log t}(\log\log t)^{-3/2-\varepsilon}} > 0$$

Tracer particle in incompressible turbulent flow

$$\mathrm{div}(\omega)(x) = \partial_{x_1}\omega_1(x) + \partial_{x_2}\omega_2(x) = -\partial^2_{x_1\, x_2} h(x) + \partial^2_{x_2\, x_1} h(x) = 0$$

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Key quantity: energy spectrum of the field

$$e(p) = \sum_{a=1}^{d} \int_{\mathbb{R}^d} e^{ipx}\mathbb{E}[\omega_a(0)\omega_a(x)]dx$$

and “Péclet number”

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Long literature in short:

- if $P(0) < \infty$ (H_{-1} condition): normal diffusion, quenched CLT (Komorowski-Olla '01, and Kozma-Tóth '17, Tóth '18 for RW $\rightsquigarrow$ Balint's talk)
- if $P(\rho) \overset{\rho\to 0^+}{\sim} \rho^{-a}$, $a > 0$, then $\int_0^T \mathbf{E}(|X(t)|^2)dt \sim T^{1+\nu}$, $\nu > 1$ (super-diffusivity) (Komorowski-Olla '02)
- our case: $P(\rho) \overset{\rho\to 0^+}{\sim} \log(1/\rho)$: boundary between diffusion and super-diffusion

Logarithmic super-diffusivity in 2d systems

Logarithmic corrections to diffusivity expected in 2d (self)-interacting diffusions:

- ASEP on $\mathbb{Z}^2$:

$$D_{\mathrm{bulk}}(t) := \frac{1}{t} \sum_{x \in \mathbb{Z}^2} |x|^2 \mathbf{E}[\eta_x(t); \eta_0(0)] \approx (\log t)^{2/3}$$

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- The 2-dimensional Anisotropic KPZ equation $\rightsquigarrow$ Giuseppe's talk

$$(\log t)^{\delta} \lesssim D_{\rm bulk}(t) \lesssim (\log t)^{1-\delta}, \qquad \delta > 0$$

(Cannizzaro-Erhard-F.T. '20)$\rightsquigarrow$ $\delta = 1/2$ (Cannizzaro-Erhard-F.T. '21$^+$)

The environment process

The environment seen by the particle

ω is divergence-free and ergodic $\Longrightarrow$ environment $\omega_t(\cdot) := \omega(X_t + \cdot)$ has law $\mathbb{P}$ (stationary).

Let $\mathcal{L}$ be generator of Markov process $(\omega_t)_{t\geq 0}$. Non-reversible!

$$\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{symmetric}} + \underbrace{\mathcal{A}}_{\text{anti-symmetric}}, \qquad \mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-, \quad \mathcal{A}_- = -\mathcal{A}_+^*$$

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Starting point: (recall: $dX_t = \omega(X_t)dt + dB_t, \; X_0 = 0$)

$$X_1(t) = B_1(t) + \int_0^t \varphi(\eta_s)ds, \qquad \varphi(\omega) := \omega_1(0).$$

Identity:

$$\int_0^\infty e^{-\lambda t}\mathbf{E}\Big[\Big(\int_0^t \varphi(\eta_s)ds\Big)^2\Big]dt = \frac{2}{\lambda^2}\mathbb{E}[\varphi(\omega)(\lambda - \mathcal{L})^{-1}\varphi(\omega)].$$

The structure of the generator

Wiener chaos analysis

- $\rightsquigarrow$ $L^2(\mathbb{P}) = \bigoplus_{n\geq 0} \mathscr{H}_n$ s.t. $\mathscr{H}_n \perp \mathscr{H}_m$ for $m \neq n$ (span of Wick products $:\hat{h}(p_1)\dots\hat{h}(p_n):$)
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Generator

- $\rightsquigarrow$ recall: ω_t is Markov and its generator $\mathcal{L}$ is such that

$$\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{symmetric}} + \underbrace{\mathcal{A}_+ + \mathcal{A}_-}_{\text{antisymmetric}}$$

- $\rightsquigarrow$ $\mathcal{L}_0, \mathcal{A}_+, \mathcal{A}_-$ satisfy for all n

$$\underbrace{\mathcal{L}_0 : \mathscr{H}_n \to \mathscr{H}_n}_{\text{Laplacian}} \quad \underbrace{\mathcal{A}_+ : \mathscr{H}_n \to \mathscr{H}_{n+1}}_{\text{creation}} \quad \underbrace{\mathcal{A}_- : \mathscr{H}_n \to \mathscr{H}_{n-1}}_{\text{annihilation}}$$

NB: $\mathcal{L}_0$ diagonal in chaos and momentum space. $\mathcal{A}_\pm$ is neither.

The truncated generator equation

- Ideally: solve $(\lambda - \mathcal{L})f = \varphi$ and then $\mathbb{E}[\varphi(\omega)(\lambda - \mathcal{L})^{-1}\varphi(\omega)] = \mathbb{E}[\varphi(\omega)f_1(\omega)]$.

  Problem: φ in first Wiener chaos $\mathscr{H}_1$, but f has components in all $\mathscr{H}_n$.

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- Idea: truncation (Landim-Quastel-Salmhofer-Yau '04)

$$\mathcal{L} \longrightarrow \mathcal{L}_n \stackrel{\text{def}}{=} \mathrm{P}_n\,\mathcal{L}\,\mathrm{P}_n, \qquad P_n : L^2(\eta) \to \bigoplus_{j \le n} \mathscr{H}_j$$

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- Advantage I: bounds

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- Advantage II:

$$(\lambda - \mathcal{L}_n)f^{(n)} = \varphi \quad \text{is finite linear system}$$

The implementation

The equation $(\lambda - \mathcal{L}_n) f^{(n)} = \varphi$ has "explicit" solution:

$$f_1^{(n)} = (\lambda - \mathcal{L}_0 + H_n)^{-1} \varphi,$$

but H_n defined only iteratively:

$$H_1 \equiv 0, \quad H_n = \mathcal{A}_+^* (\lambda - \mathcal{L}_0 + H_{n-1})^{-1} \mathcal{A}_+.$$

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Let $\mathcal{E}_k(x) := \sum_{j\le k} \frac{1}{j!}\left(\frac{1}{2}\log\log(1/x)\right)^j$. NB: $\mathcal{E}_k(x) \stackrel{k\to\infty}{\to} \sqrt{\log(1/x)}$

Theorem (very rough version)

Operator bounds

$$H_{2n} \lesssim (-\mathcal{L}_0)\mathcal{E}_n(-\mathcal{L}_0), \quad H_{2n+1} \gtrsim (-\mathcal{L}_0)\frac{\log[(-\mathcal{L}_0)^{-1}]}{\mathcal{E}_n(-\mathcal{L}_0)}$$

Some crucial issues

- Iterative proof: UB on $H_2 \to$ LB on $H_3 \to$ UB on H_4...

(recall: $H_1 \equiv 0, \quad H_n = \mathcal{A}_+^*(\lambda - \mathcal{L}_0 + H_{n-1})^{-1}\mathcal{A}_+.$)

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- Why $(\log t)^{2/3}$ for 2d-ASEP and $(\log t)^{1/2}$ for the diffusion in the curl of the GFF? Anisotropic vs. Isotropic dispersion relation
- structural, not technical, difference:
  - for 2d-ASEP [Yau '04] at step n, $\log^{2/3-\nu_n}$ bound, with
    $$\nu_n = O(\exp(-n)).$$
  - in our case, only loglog gain at each step

Summary and open questions

We looked at the SDE

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(diffusion in a divergence-free, two-dimensional random field)

- result: logarithmic super-diffusivity $\mathbf{E}(|X(t)|^2) \approx t\sqrt{\log t}$ (conjectured by Tóth-Válko, 2012)
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- interplay between diffusion (dB_t) and motion on level-lines of GFF.

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