Multi-index based regularity structures: Stochastic estimate of the model

Pablo Linares, Markus Tempelmayr, Pavlos Tsatsoulis

based on work with Jonas Sauer, Scott Smith, Hendrik Weber

Max Planck Institute for Mathematics in the Sciences, Leipzig

version 08.09.2021 after talk

Quasi-linear parabolic equations ...

Quasi-linear parabolic equation, simplest representative:

$(\partial_2 - \partial_1^2)u = a(u)\partial_1^2 u + \xi \quad \text{in } \mathbb{R}^2.$

Rather treat as elliptic: x_1 space-like, x_2 time-like.

Degree of singularity same as for gPAM

$(\partial_2 - \partial_1^2)u = a(u)\xi \quad \text{in } \mathbb{R}^2.$

If driver ξ = space-time white noise, SPDE is singular:

$a(u) \in C^\alpha$ and $\xi, \partial_1^2 u \in C^{\alpha-2}$ for $\alpha = \frac{1}{2}-$.

... as training ground for regularity structures

An active area

O.& Weber: *Quasi-linear SPDEs via rough paths*
parametric Ansatz, controlled rough path, $\alpha \in (\frac{2}{3}, 1)$

Furlan & Gubinelli: *Para-controlled quasilinear SPDEs*
parametric Ansatz, para-composition, $\alpha \in (\frac{2}{3}, 1)$

Bailleul & Debussche & Hofmanova:
Quasilinear generalized parabolic Anderson model equation
para-products, $\alpha \in (\frac{2}{3}, 1)$

Gerencsér & Hairer / Gerencsér:
A solution theory for quasilinear singular SPDEs / Nondivergence form quasilinear heat equations driven by space-time white noise
parametric Ansatz, integral equation, regularity structures, $(\alpha \in (\frac{1}{2}, \frac{2}{3})/(\frac{2}{5}, \frac{1}{2}))$

O. & Sauer & Smith & Weber:
parametric Ansatz, regularity structures, $(\alpha \in (\frac{1}{2}, \frac{2}{3})/\alpha \in (0, 1))$

The three tasks in Hairer et. al. 's approach

build regularity structure $(\mathsf{T}, \mathsf{A}, \mathsf{G})$;

purely algebraic.

For us: representation /Lie theory

arXiv Linares & Tempelmayr.

construct model (Π_x, Γ_{xy});

stochastic estimates.

For us: first-order Malliavin calculus.

Today & Friday.

solve by modelled distribution;

purely analytic.

Schauder theory

arXiv Sauer & Smith & Weber.

... without trees in all three tasks

Our result paraphrased

T*: direct product over index set $\{\beta\}$,

Π_x: random function with values in T*,

Γ_{xy}: random function with values in G* $\subset$ End(T*),

$\Gamma_{xy} -$ id strictly triangular w. r. t. homogeneity $|\beta| \in$ A.

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p \le C(\beta,p)|y-x|^{|\beta|},$$

$$\mathbb{E}^{\frac{1}{p}}|(\Gamma_{xy})_\beta^\gamma|^p \le C(\beta,p)|y-x|^{|\beta|-|\gamma|}.$$

Note "annealed norm" — Hölder space-time norm outside, stochastic L^p-norm $\mathbb{E}^{1/p}|\cdot|^p$ inside.

Note whole space-time i. e. $x,y \in \mathbb{R}^2$ — infra-red part of estimate enforces uniqueness, amounts to BPHZ choice of renormalization.

Scaling as guiding principle. Automated proof.

Our approach to renormalization ...

Renormalized PDE: $(\partial_2 - \partial_1^2)u + h = a(u)\partial_1^2 u + \xi$.

Postulates on counter term, a priori of form $h[a,\xi,u](y)$:

* *local* in solution u: $h[a,\xi](y,u(y))$,

* *shift invariance in y-space* $\Longleftrightarrow$ deterministic: $h[a](u(y))$,

* *shift covariance in u-space*:

$$h[a(v+\cdot)](u) = h[a](v+u) \text{ for all } v \in \mathbb{R}.$$

Counter term given by functional $h[a]$ via $h[a(u(y)+\cdot)]$.

Choice of origin in u and y-space has no effect on renormalization.

... top-down and analytic

Our approach to the model

Model is to parameterize solution manifold $u = u[a, \xi](y)$.

Shift in u-space:
$\tilde{u} = u[a(v + \cdot), \xi](y) + v$ is again a solution.

Still impoverished, hence pass to: $u = u[a, p, \xi](y)$
with p space-time polynomial *mod constants*,
anchored by $u[0, p' + p, \xi] = u[0, p', \xi] + p$,
collateral: equation only satisfied mod polynomials.

Coordinates for a: $\mathsf{z}_k[a] := \frac{1}{k!}\frac{d^k a}{du^k}(0)$ for $k \geq 0$,
Coordinates for p: $\mathsf{z}_n[p] := \frac{1}{n!}\frac{d^n p}{dy^n}(0)$ for $n = (n_1, n_2) \neq 0$.
Take partial derivatives w. r. t. $\{\mathsf{z}_k\}_k$, $\{\mathsf{z}_n\}_n$,
naturally leads to multi-index $\beta = \{\beta(k)\} \cup \{\beta(n)\}$.

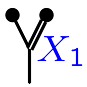
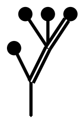
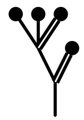
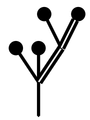
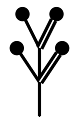
Identification of model

PDE hierarchy for partial derivatives $(\Pi_\beta[\xi](y), c_\beta)$, e.g.

$$(\partial_2\text{-}\partial_1^2)\Pi_{e_2+e_{(1,0)}} = 2\Pi_{e_{(1,0)}}\Pi_0\partial_1^2\Pi_0 - 2\Pi_{e_{(1,0)}}c_{e_1},$$
$$(\partial_2\text{-}\partial_1^2)\Pi_{e_1+e_2} = \Pi_0\partial_1^2\Pi_{e_2}+\Pi_{e_2}\partial_1^2\Pi_0+\Pi_0^2\partial_1^2\Pi_{e_1}+2\Pi_0\Pi_{e_1}\partial_1^2\Pi_0$$
$$-(c_{e_1+e_2} + 2\Pi_{e_1}c_{e_1} + \Pi_0 c_{e_0+e_2} + 4\Pi_0 c_{2e_1} + 3\Pi_0^2 c_{e_0+e_1}).$$

Have $\Pi_{e_n}(y) = y^n$, $c_{e_n} = 0$ (polynomial sector),
otherwise population condition $\sum_k k\beta(k) \geq \sum_n \beta(n)$.

Multi-indices β top-down and analytic,
trees τ bottom-up and combinatorial. Dictionary:

$e_2 + e_{(1,0)} \mathrel{\hat{=}} 2$  $\quad e_1 + e_2 \mathrel{\hat{=}}$  $+$  $+$  $+ 2$ .

An algebra structure ...

Combine to $\Pi[\xi](y)$ and c with values in
formal power series algebra $\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_n]]$.

PDE $(\partial_2 - \partial_1^2)\Pi = \Pi^-$ up to polynomials,

where $\Pi^- := \xi 1 + \sum_k \mathsf{z}_k \Pi^k \partial_1^2 \Pi - \sum_k \frac{1}{k!} \Pi^k (D^{(0)})^k c$,

where $D^{(0)} := \sum_{k\geq 0} (k+1)\mathsf{z}_{k+1}\partial_{\mathsf{z}_k}$ generates u-shift.

Postulates on counter term translate into axioms on c:

Local: $c_\beta = 0$ unless $\beta(n) = 0$ for all $n \neq 0$,
Invariant w.r.t. y-**shift**: c independent on y and ξ,
Covariant w.r.t. u-**shift**: c feeds in via its generator $D^{(0)}$.

... provides a compact formulation

Scale invariance motivates homogeneity

Covariance under **space-time shift** by $z \in \mathbb{R}^2$:

$$u[a, p(z+\cdot), \xi(z+\cdot)](y) = u[a, p, \xi](z+y).$$

On level of Π: $\Pi_\beta[\xi(z+\cdot)](y) = \Pi_\beta[\xi](z+y)$.

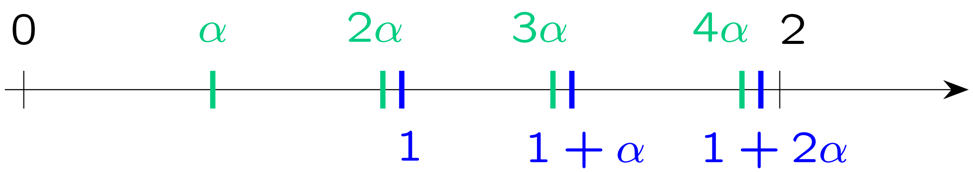
Covariance under **space-time scaling** by $s \in (0, \infty)$:

$$u[a, s^{-\alpha} p(S\cdot), s^{2-\alpha}\xi(S\cdot)](y) = s^{-\alpha} u[a(s^{-\alpha}\cdot), p, \xi](Sy).$$

On level of Π: $\Pi_\beta[\xi](Sy) = s^{|\beta|}\Pi_\beta[s^{2-\alpha}\xi(S\cdot)](y)$,

provided $|\beta| = \alpha(1 + \sum_k k\beta(k) - \sum_n \beta(n)) + \sum_n |n|\beta(n)$,

where $|n| := n_1 + 2n_2$.



Two further invariances ...

Under **reflection of space**: For $Rx = (-x_1, x_2)$ we have

$$u[a, p(R\cdot), \xi(R\cdot)](y) = u[a, p, \xi](Ry).$$

On level of (Π, c): $\Pi_\beta[\xi(R\cdot)](y) = (-1)^{\sum_n n_1 \beta(n)} \Pi_\beta[\xi](Ry)$
and $c_\beta = 0$ for $\sum_n n_1 \beta(n)$ odd.

Under **shift of a-space**: For any $z_0 > -1$
$u[z_0 + a, p, \xi]$ satisfies same equation as u
with $\partial_2 - \partial_1^2$ replaced by $\partial_2 - (z_0 + 1)\partial_1^2$.
On level of (Π, c): convergent power series in z_0 and
$(\Pi_{\beta + l e_0}, c_{\beta + l e_0}) = \frac{1}{l!}\frac{d^l}{dz_0^l}|_{z_0 = 0}(\Pi_\beta, c_\beta)$
for all β with $\beta(k = 0) = 0$.

... crucial for renormalization

Assumption I on law $\mathbb{E}$ of driver ξ ...

Recall shift covariance $\Pi_\beta[\xi(z+\cdot)](y)=\Pi_\beta[\xi](z+y)$

and reflection covariance $\Pi_\beta[\xi(R\cdot)](y)=(-1)^{\sum_n n_1\beta(n)}\Pi_\beta[\xi](Ry)$.

Assumption: $\mathbb{E}$ is invariant under shift & reflection.

Recall $\Pi^- := \xi 1 + \sum_k \mathsf{z}_k \Pi^k \partial_1^2 \Pi - \sum_k \frac{1}{k!}\Pi^k (D^{(0)})^k c$.

By shift invariance may inductively choose c_β s. t.

$\mathbb{E}\Pi^-_\beta(y)=0$ for all y and β with $\beta(n)=0$.

By reflection invariance have

$\mathbb{E}\Pi^-_\beta(y)=0$ for all y and β with $\sum_n n_1\beta(n)$ odd.

In particular achieve $\mathbb{E}\Pi^-_\beta(y)=0$ for all y and $|\beta|<2$.

... amounts to BPHZ choice of renormalization

From BPHZ choice of renormalization ...

Recall: $\forall\ \beta$ with $\beta(n)=0$ for all $n\neq 0$, $\beta(k=0)=0$, and $|\beta|<2$
choose deterministic $c_\beta(z_0)$ s. t. $\mathbb{E}\Pi^-_\beta(z_0,y)=0$ for all y .

Set of constants $\{c_\beta\}$ determines counter term h:

$$h(u)=h[a(u+\cdot)]=\sum_\beta c_\beta(a(u))\prod_{k\geq 1}(\tfrac{1}{k!}\tfrac{d^k a}{du^k}(u))^{\beta(k)},$$

where sum is over all β
with $\beta(n)=0$ for all $n\neq 0$, $\beta(k=0)=0$, and $|\beta|<2$.

Counter terms naturally indexed by fewer β's (7)
rather than larger number of trees τ (15 for $\alpha=\frac{1}{2}-$).

... to counter term

Assumption II on law $\mathbb{E}$ of driver ξ

Assumption: Spectral gap, i. e. estimate of variance

$$\mathbb{E}(F-\mathbb{E}F)^2 \le \mathbb{E}\|\frac{\partial F}{\partial \xi}\|^2 \quad \text{for all } F = F[\xi],$$

where $\frac{\partial F}{\partial \xi}$ is functional/Malliavin derivative of F, i. e.

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(F[\xi+\epsilon\delta\xi]-F[\xi]) = \int_{\mathbb{R}^2} dy\,\delta\xi(y)\,\frac{\partial F}{\partial \xi}[\xi](y) \quad \text{for } \delta\xi = \delta\xi(y),$$

with $\|\cdot\|$ = fractional Sobolev norm $\dot H_2^{\frac{1}{2}-\alpha}$:

$$\|u\|^2 := \int_{\mathbb{R}^2} dy |(\partial_1^4 - \partial_2^2)^{\frac{1}{4}(\frac{1}{2}-\alpha)} u|^2 \quad \text{for } u = u(y).$$

Fits BPHZ choice of renormalization
since variance compensates expectation.

Example:
Centered Gaussian with Cameron-Martin norm $\ge \|\cdot\|_*$.

Postulates on centered model (Π_x, Π_x^-, c), Γ_{xy}

Generic base point x. As before:

* $(\partial_2 - \partial_1^2)\Pi_{x\beta} = \Pi_{x\beta}^-$ mod polynomials of degree $\leq |\beta| - 2$.
* $c_\beta = 0$ unless $|\beta| < 2$ and $\beta(n) = 0$ for all $n \neq 0$.
* $\Pi_x^- = \xi 1 + \sum_{k\geq 0} \mathsf{z}_k {\Pi_x}^k \partial_1^2 \Pi_x - \sum_{k\geq 0} \frac{1}{k!} {\Pi_x}^k (D^{(\mathbf{0})})^k c$.

In addition centering: $\partial^n \Pi_{x\beta}(x) = 0$ for $|n| < |\beta|$.

Generic pair of base points (x, y).

$\Pi_x^- = \Gamma_{xy} \Pi_y^-$ and $\Pi_x = \Gamma_{xy}\Pi_y + \Pi_x(y)$

plus usual other axioms on Γ.

Full statement of main result

Assumptions I (shift and reflection) & II (spectral gap) on $\mathbb{E}$.
Assumptions on $\alpha \notin \mathbb{Q}$: $\alpha \in (\frac{1}{4}, \frac{1}{2})$. Then for all $x, y \in \mathbb{R}^2$

$$\mathbb{E}^{\frac{1}{p}}|\Pi^{-}_{x\beta t}(x)|^p \leq C(\beta, p)(\sqrt[4]{t})^{|\beta|-2} \quad \text{for all } t \in (0, \infty),$$

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p \leq C(\beta, p)|y - x|^{|\beta|} \quad \text{for all } y \in \mathbb{R}^2,$$

$$\mathbb{E}^{\frac{1}{p}}|(\Gamma_{xy})_{\beta}{}^{\gamma}|^p \leq C(\beta, p)|y - x|^{|\beta|-|\gamma|}.$$

Here $(\cdot)_t$ denotes the application of
the semi-group generated by $(\partial_1^4 - \partial_2^2)$ and amounts to
a mollification on length scale $\sqrt[4]{t}$ consistent with scaling.

... both construction and estimate

Malliavin derivative gains in modeledness

From $\xi \in C^{\alpha-2}$ to $\delta\xi \in \dot{H}_2^{\alpha-\frac{1}{2}}$ (Cameron-Martin):
gain of $\frac{3}{2}$ derivatives (but loss from C_0-scale to L_2-scale)

Infinitesimal perturbation $\delta\xi$ of ξ generates
infinitesimal perturbation $\delta\Pi_x$ and $\delta\Pi_x^-$.

From Π_x to $\delta\Pi_x$ no plain gain of $\frac{3}{2}$ derivatives,
but just in terms of **controlled rough path condition**

$$\mathbb{E}^{\frac{1}{q}}|(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}\Pi_z)_\beta(y)|^q$$
$$\lesssim |y-z|^{\alpha+\frac{3}{2}-}(|y-z|+|z-x|)^{|\beta|-\alpha}\times(\text{weighted norm of }\delta\xi)$$

for some modelled distribution $\mathrm{d}\Gamma_{xz}$ (z secondary base point).

Modeledness enables reconstruction

Recall $\Pi_x^- = \xi 1 + \sum_k \mathsf{z}_k (\Pi_x)^k \partial_1^2 \Pi_x - \sum_k \frac{1}{k!} (\Pi_x)^k (D^{(0)})^k c$.

For rough-path increments of Malliavin derivative:

$$\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}\Pi_z^- = \delta\xi 1 + \textstyle\sum_{k\geq 0} \mathsf{z}_k (\Pi_x)^k \partial_1^2 (\delta\Pi_x - \mathrm{d}\Gamma_{xz}\Pi_z)$$

when evaluated at $y = z$; divergent c drops out!

Ansatz for **modelled distribution** $\mathrm{d}\Gamma$:

$$\mathrm{d}\Gamma_{xz} = \delta\pi_x(z)\Gamma_{xz}D^{(0)} + \mathrm{d}\pi_{xz}^{(1,0)}\Gamma_{xz}D^{(1,0)} \qquad \neq \delta\Gamma_{xz}$$

where $D^{(1,0)} = \partial_{z_{(1,0)}}$ with $\mathrm{d}\pi_{xz}^{(1,0)}$ to be chosen.

Satisfied continuity condition of modelled distribution:

$\mathbb{E}^{\frac{1}{q}} |(\mathrm{d}\Gamma_{xy} - \mathrm{d}\Gamma_{xz}\Gamma_{zy})_\beta^\gamma|^q$ in terms of $|y - z|^{\alpha + \frac{3}{2} -}$.

Summary of main aspects of approach

Guided by symmetries (y-scaling, u-shift, a-shift).

Analysis replaces combinatorics:
derivatives w. r. t. a ($\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_n]]$) and ξ (Malliavin)
vs. trees & diagrams.

Spectral gap naturally complements
BPHZ choice of renormalization.

Interpret Malliavin derivative of model
as modelled distribution.

Exact scaling: annealed estimates in whole space-time
encode BPHZ choice of renormalization.
Use weighted Besov norms to bridge gap between
L_2-based Malliavin and C_0-based modelled distribution calculus.

This combinatorics does not play a role

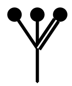
$$(\partial_2 - \partial_1^2)\Pi_0 = \xi$$

$$\Pi_0 = $$  .

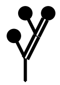
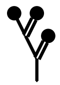
$$(\partial_2 - \partial_1^2)\Pi_{e_1} = \Pi_0 \partial_1^2 \Pi_0 - c_{e_1}$$

$$\Pi_{e_1} = $$  .

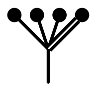
$$(\partial_2 - \partial_1^2)\Pi_{e_2} = \Pi_0^2 \partial_1^2 \Pi_0 - (c_{e_2} + \Pi_0 c_{e_1})$$

$$\Pi_{e_2} = $$  .

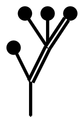
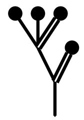
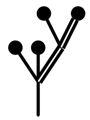
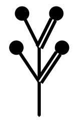
$$(\partial_2 - \partial_1^2)\Pi_{2e_1} = \Pi_0\partial_1^2\Pi_{e_1} + \Pi_{e_1}\partial_1^2\Pi_0 - (c_{2e_1} + \Pi_0 c_{e_0+e_1})$$

$\Pi_{2e_1} =$  $+$ .

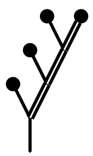
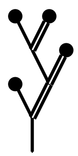
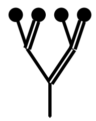
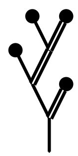
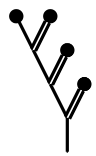
$$(\partial_2 - \partial_1^2)\Pi_{e_3} = \Pi_0^3\partial_1^2\Pi_0 - (c_{e_3} + 3\Pi_0 c_{e_2} + 3\Pi_0^2 c_{e_1})$$

$\Pi_{e_3} =$ .

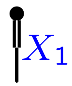
$$
\begin{aligned}
&(\partial_2 - \partial_1^2)\Pi_{e_1+e_2} \\
&= \Pi_0 \partial_1^2 \Pi_{e_2} + \Pi_{e_2}\partial_1^2 \Pi_0 + \Pi_0^2 \partial_1^2 \Pi_{e_1} + 2\Pi_0 \Pi_{e_1} \partial_1^2 \Pi_0 \\
&\quad - (c_{e_1+e_2} + 2\Pi_{e_1} c_{e_1} + \Pi_0 c_{e_0+e_2} + 4\Pi_0 c_{2e_1} + 3\Pi_0^2 c_{e_0+e_1})
\end{aligned}
$$

$\Pi_{e_1+e_2} =$  $+$  $+$  $+\,2$ 

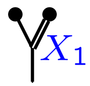
$$
\begin{aligned}
(\partial_2 - \partial_1^2)\Pi_{3e_1} &= \Pi_0 \partial_1^2 \Pi_{2e_1} + \Pi_{e_1}\partial_1^2 \Pi_{e_1} + \Pi_{2e_1} \partial_1^2 \Pi_0 \\
&\quad - (c_{3e_1} + \Pi_{e_1} c_{e_0+e_1} + \Pi_0 c_{e_0+2e_1} + \Pi_0^2 c_{2e_0+e_1})
\end{aligned}
$$

$\Pi_{3e_1} =$  $+$  $+$  $+$  $+$ .

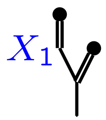
$$(\partial_2 - \partial_1^2)\Pi_{e_1+e_{(1,0)}} = \Pi_{e_{(1,0)}}\partial_1^2\Pi_0$$

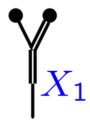
$$\Pi_{e_1+e_{(1,0)}} =$$  .

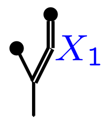
$$(\partial_2 - \partial_1^2)\Pi_{e_2+e_{(1,0)}} = 2\Pi_{e_{(1,0)}}\Pi_0\partial_1^2\Pi_0 - 2\Pi_{e_{(1,0)}}c_{e_1}$$

$$\Pi_{e_2+e_{(1,0)}} = 2$$  .

$$(\partial_2 - \partial_1^2)\Pi_{2e_1+e_{(1,0)}}$$
$$= \Pi_{e_1+e_{(1,0)}}\partial_1^2\Pi_0 + \Pi_{e_{(1,0)}}\partial_1^2\Pi_{e_1} + \Pi_0\partial_1^2\Pi_{e_1+e_{(1,0)}}$$
$$- \Pi_{e_{(1,0)}}c_{e_0+e_1}$$

$$\Pi_{2e_1+e_{(1,0)}} =$$ 

$+$ 

$+$ 

.