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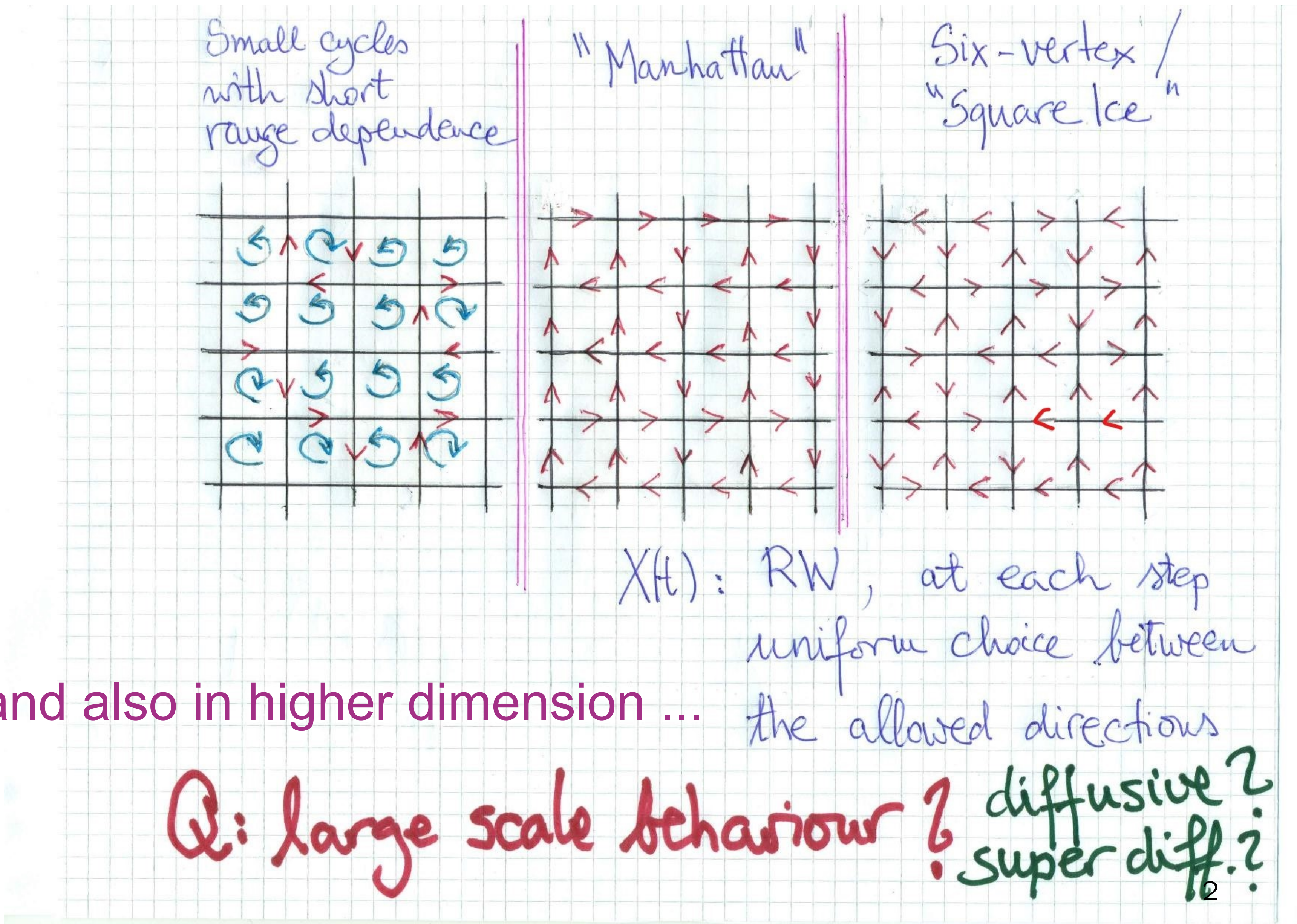
QUENCHED CLT FOR RANDOM WALK IN DIVERGENCE-FREE RANDOM DRIFT FIELD

PDE & Randomness
LMS-Bath, 2021-09-07



... and also in higher dimension ...

RW in divergence-free random drift field:

$(\Omega; \pi; \tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d)$	probab. sp., ergodic $\mathbb{Z}^d$-action
$\mathcal{E} = \{k \in \mathbb{Z}^d : \lvert k \rvert = 1\}$	possible steps of the rw
$p : \Omega \to [0,1]^{\mathcal{E}},$	jump rates of the rw

RWRE: Given $\omega \in \Omega$, $t \mapsto X(t) \in \mathbb{Z}^d$ cont. time Markov chain:

$$\mathbf{P}_\omega \left(X(t+dt) = x + k \middle| X(t) = x \right) = p_k(x,\omega)dt = p_k(\tau_x \omega)dt.$$

Separate symmetric and skew-symmetric part of jump rates:

"conductances" : $\quad s_k(\omega) := \dfrac{p_k(\omega) + p_{-k}(\tau_k \omega)}{2} = s_{-k}(\tau_k \omega),$

"flow" : $\quad v_k(\omega) := \dfrac{p_k(\omega) - p_{-k}(\tau_k \omega)}{2} = -v_{-k}(\tau_k \omega).$

Assumptions:

$$\sum_{k\in\mathcal{E}} p_{-k}(\tau_k\omega) \equiv \sum_{k\in\mathcal{E}} p_k(\omega) \;\Leftrightarrow\; \sum_{k\in\mathcal{E}} v_k(\omega) \equiv 0 \qquad \pi\text{-a.s.} \qquad (\texttt{DFR})$$

$$s_k(\omega) \geq s_* > 0, \qquad \pi\text{-a.s.} \qquad (\texttt{ELL})$$

$$\int_\Omega v_k(\omega)d\pi(\omega) = 0 \qquad (\texttt{NODRIFT})$$

In the examples

$$s_k(\omega) \equiv \frac{1}{2d}, \qquad v_k(\omega) = 0, \pm\frac{1}{2d}$$

Analogous diffusion problem: **diffusion in divergence-free random drift field** $t \mapsto X(t) \in \mathbb{R}^d$ with infinitesimal generator

$$L := \frac{1}{2}\nabla \cdot a\nabla + v \cdot \nabla$$

$a(\omega, \cdot) = a(\omega, \cdot)^\dagger : \mathbb{R}^d \to \mathbb{R}^{d\times d}$; $v(\omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are stat+erg,

$$0 < a_* I_{d\times d} \leq a(\cdot, \omega) \leq a^* I_{d\times d} < \infty \qquad \pi\text{-a.s.} \qquad (\texttt{ELL})$$

$$\operatorname{div} v(\cdot, \omega) \equiv 0 \qquad \pi\text{-a.s.} \qquad (\texttt{DFR})$$

$$\int_\Omega v(\cdot, \omega) d\pi(\omega) = 0 \qquad (\texttt{NODRIFT})$$

$$dX(t) = a(\omega, X(t))^{1/2} dB(t) + \left(\frac{1}{2}\nabla a(\omega, X(t)) + v(\omega, X(t))\right) dt,$$

with $a \equiv I_{d\times d}$: $\qquad dX(t) = dB(t) + v(\omega, X(t))dt$

The (H_{-1})-Condition:

Let

$$C_{kl}(x) := \int_\Omega v_k(\omega) v_l(\tau_x \omega) d\pi(\omega), \qquad \widehat{C}_{kl}(p) := \sum_{x \in \mathbb{Z}^d} e^{\sqrt{-1} p \cdot x} C_{kl}(x)$$

An infrared integrability condition on the correlations:

$$\int_{[-\pi,\pi]^d} \frac{\sum_{k \in \mathcal{E}} \widehat{C}_{kk}(p)}{\sum_{i=1}^d (1 - \cos(p_i))} dp < \infty \qquad (H_{-1})$$

The natural analytic setting:

$$\Delta : \mathcal{L}^2(\Omega, \pi) \to \mathcal{L}^2(\Omega, \pi) : \qquad \Delta f(\omega) := \sum_{k \in \mathcal{E}} (f(\tau_k \omega) - f(\omega))$$

$$\mathcal{H}_{-1} := \mathtt{Ran}(|\Delta|^{1/2}) = \mathtt{Dom}(|\Delta|^{-1/2}) \subsetneq \mathcal{L}^2(\Omega, \pi)$$

What is the meaning of the (H_{-1}) condition?

$$\mathbf{E}_\omega\left(dX(t)|X(t)=x\right)=\underbrace{\sum_{k\in\mathcal{E}} k s_k(\tau_x\omega)\,dt}_{\psi(\tau_x\omega)\,\in\,\mathbb{R}^d}+\underbrace{\sum_{k\in\mathcal{E}} k v_k(\tau_x\omega)\,dt}_{\varphi(\tau_x\omega)\,\in\,\mathbb{R}^d}$$

ψ is a gradient:

$$\psi_i(\omega)=s_{e_i}(\omega)-s_{e_i}(\tau_{-e_i}\omega)\in\mathcal{H}_{-1}$$

The assumption (H_{-1}) is equivalent to: (1) *Analitically*

$$\varphi\in\mathcal{H}_{-1}. \qquad (H_{-1})$$

(2) *Probabilistically:* Let $t\mapsto S(t)$ be SSRW *independent of* ω

$$\lim_{T\to\infty} T^{-1}\int_\Omega \mathbf{E}\left(\left|\int_0^T \varphi(\tau_{S(t)}\omega)dt\right|^2\right)d\pi(\omega)<\infty. \qquad (H_{-1})$$

First consequences of the assumptions?

- (DFR) $\Rightarrow$ The *environment process*:

$$t \mapsto \eta_t := \tau_{X(t)}\omega$$

is stationary and ergodic Markov process in (Ω, π).

- (H_{-1}) $\Rightarrow$ (NODRIFT) $\Rightarrow$ zero overall drift of the walk:

$$\mathbf{E}\left(X(t)\right) := \int_\Omega \mathbf{E}_\omega\left(X(t)\right) d\pi(\omega) = 0.$$

- Thus, (DFR) & (NODRIFT) $\Rightarrow$ (by the ergodic thm) LLN:

$$t^{-1}X(t) \to 0, \quad \text{a.s.}$$

- (ELL) & (H_{-1}) $\Rightarrow$ diffusive bounds (not straightforward):

$$0 < \varliminf_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) \le \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) < \infty.$$

Question left open: CLT?

Why care?

(1) Genuinely **more general** than RW among random conductances. Even the $s \equiv 1$ case is interesting and non-trivial.

(2) **Physical motivation**: Diffusion of suspended particles in *incompressible turbulent flow*, in stationary regime.

(3) Assuming a.c. stationary measure $\varrho(\omega)d\pi(\omega)$ for $t \mapsto \eta(t) \in \Omega$, a random change of time leads to RW in div-free drift field.

Why different?

Essentially (but not only) due to **non-self-adjointness.**

History

From Lucretius Carus (cca 60 BC) ...... to ......
G Papanicolaou, SRS Varadhan (1981) ...... (more later)

From “Brownian Motion” at

History[edit source]

The Roman philosopher-poet Lucretius' scientific poem "On the Nature of Things" (c. 60 BC) has a remarkable description of the motion of dust particles in verses 113–140 from Book II. He uses this as a proof of the existence of atoms:

……………………………

Although the mingling motion of dust particles is caused largely by air currents, the glittering, tumbling motion of small dust particles is caused chiefly by true Brownian dynamics; Lucretius "perfectly describes and explains the Brownian movement by a wrong example".[8]

I don't think it is a "wrong example" of BM.
It shows universality at very different length-scales.

Titus Lucretius Carus: De Rerum Natura, Liber Secundus (cca 60 BC)

sic a principiis ascendit motus et exit // paulatim nostros ad sensus, ut moveantur // illa quoque, in solis quae lumine cernere quimus // nec quibus id faciant plagis apparet aperte.

Thus motion ascends from the primevals on, // And stage by stage emerges to our sense, // Until those objects also move which we // Can mark in sunbeams, though it not appears // What blows do urge them.

(Transl. William Ellery Leonard, 1916)

Proposition ("Helmholtz's theorem"). *Let* $v_k(x,\omega) := v_k(\tau_x\omega)$ *be an ergodic,* $\mathcal{L}^2$*, divergence-free flow on* $\mathbb{Z}^d$ *(as before).*
(i) $\exists!$ *tensor-field* $x \mapsto h(x,\cdot) \in \mathcal{L}^2(\Omega \to \mathbb{R}^{\mathcal{E}\times\mathcal{E}}, \pi)$ *such that*

`cocycle` $h_{k,l}(y,\omega) - h_{k,l}(x,\omega) = h_{k,l}(y-x,\tau_x\omega) - h_{k,l}(0,\tau_x\omega),$

`tensor` $h_{l,k}(x,\omega) = h_{-k,l}(x+k,\omega) = h_{k,-l}(x+l,\omega) = -h_{k,l}(x,\omega),$

$\texttt{v} = \texttt{rot}\,\texttt{h}$ $v_k(x,\omega) = \sum_{l\in\mathcal{E}} h_{k,l}(x,\omega).$

($x \mapsto h(x)$ *is the* stream tensor*.)*

(ii) (H_{-1}) *is valid (for* v*) if and only if for some* $h_{k,l} \in \mathcal{L}^2(\Omega,\pi)$

$$h_{k,l}(x,\omega) = h_{k,l}(\tau_x,\omega).$$

[Comment: uniqueness in (i) = Coulomb gauge]

Theorem. *Assume* (DFR), (ELL), (H_{-1}). *The non-degenerate covariance matrix* $(\sigma^2)_{ij} := \lim_{T\to\infty} T^{-1}\mathbf{E}\left(X_i(T)X_j(T)\right)$ *exists. For any* $m \in \mathbb{N}$, $t_1, \ldots, t_m \in \mathbb{R}_+$ *and* $f : \mathbb{R}^{md} \to \mathbb{R}$ *cont & bdd*

(i) [G Kozma, BT (*Ann. Probab.* 2017)]
CLT in probability w.r.t. the environment – *semi-quenched*:

$$\mathbf{E}_\omega\left(f(..., T^{-1/2}X(Tt_j), ...)\right) \overset{\pi\text{-prob}}{\longrightarrow} \mathbf{E}\left(f(..., W_\sigma(t_j), ...)\right).$$

(ii) [BT (*Ann. Probab.* 2018)]
CLT almost surely w.r.t. the environment – *quenched*:
If the marginally stronger integrability condition

$$\int_\Omega \left|h_{k,l}(\omega)\right|^{2+\varepsilon} d\pi(\omega) < \infty \qquad \text{(H-1-turbo)}$$

holds, then

$$\mathbf{E}_\omega\left(f(..., T^{-1/2}X(Tt_j), ...)\right) \overset{\pi\text{-a.s.}}{\longrightarrow} \mathbf{E}\left(f(..., W_\sigma(t_j), ...)\right).$$

Historical comments (sketchy, far from complete):

[SM Kozlov (1979)], [G Papanicolaou, SRS Varadhan (1981)]:
$v \equiv 0$, self-adjoint, diffusion, initiation of the problem

[H Osada (1983)], [SM Kozlov (1985)]:
$h \in \mathcal{L}^\infty$, O: quenched diffusion; K: semi-quenched walk

[K Oelschläger (1988)], [M Avellaneda, A Majda (1991)], [A Fannjiang, G Papanicolaou (1996)]:
$s = \text{const.}$, $h \in \mathcal{L}^2$, semi-quenched, diffusion, some restrictions

[A Fannjiang, T Komorowski (1997)]:
$s = \text{const.}$, $h \in \mathcal{L}^{d+\varepsilon}$, quenched diffusion.

[T Komorowski, S Olla (2003)], [J-D Deuschel, H Kösters (2008)]:
$h \in \mathcal{L}^\infty$, KO: semi-quenched walk, DK: quenched walk

[T Komorowski, C Landim, S Olla (2012)]:
$h \in \mathcal{L}^d$, semi-quenched walk, + diffusion with s, h Gaussian

Elements of proof:

- **Quenched tightness:** Nash moment bound
  extended from $h \in \mathcal{L}^\infty(\Omega, \pi)$ to $h \in \mathcal{L}^{2+\varepsilon}(\Omega, \pi)$:

$$\overline{\lim_{t\to\infty}}\, t^{-1/2}\mathbf{E}_\omega\left(|X(t)|\right) \le C < \infty, \qquad \pi\text{−a.s.}$$

- **Harmonic coordinates:** Kozlov, Osada . . .
  extended from $h \in \mathcal{L}^\infty(\Omega, \pi)$ to $h \in \mathcal{L}^2(\Omega, \pi)$:
  Given $\varphi \in \mathcal{H}_{-1}$,
  find $\theta \in \mathcal{L}^2(\Omega \to \mathbb{R}^{\mathcal{E}}, \pi)$, s.t. $\theta_k(\omega) + \theta_{-k}(\tau_k\omega) = 0$

$$\sum_{k\in\mathcal{E}} p_k(\omega)\theta_k(\omega) = \varphi,$$

$$\theta_k(\omega) + \theta_l(\tau_k\omega) = \theta_l(\omega) + \theta_k(\tau_l\omega)$$

  Let $x \mapsto \Theta(x, \omega)$ be the $\mathbb{Z}^d$-cocycle with $\operatorname{grad}\Theta(x, \omega) = \theta(\tau_x\omega)$

Elements of proof (ctd):

∘ **"Relaxed Sector Condition" (RSC):** Prove that

$|\Delta|^{-1/2}(L - L^*)|\Delta|^{-1/2}$ is **skew-self-adjoint** over $\mathcal{L}^2(\Omega, \pi)$.

∘ **Quenched martingale CLT** for

$$t \mapsto X(t) - \Theta(X(t), \omega)$$

(given the harmonic coordinates, this comes for free)

∘ **Control the corrector:**

$$\lim_{t\to\infty} \mathbf{P}_\omega \left(t^{-1/2} |\Theta(X(t), \omega)| > \delta \right) = 0, \qquad \pi\text{- a.s.}$$

relying on quenched tightness and [Zygmund (1951)]'s **unrestricted a.s. ergodic theorem** (over $\mathbb{Z}^d$).

RSC – functional analytic details: (assume $s \equiv 1$, a.s.)
Some **operators** on the Hilbert space $\mathcal{L}^2(\Omega,\pi)$:

$$\begin{array}{lll} \mathcal{L}^2(\Omega,\pi)\text{-grad}: & \nabla_k f(\omega) := f(\tau_k\omega) - f(\omega), & \nabla_k^* = \nabla_{-k} \\ \mathcal{L}^2(\Omega,\pi)\text{-Lapl}: & \Delta f(\omega) := \sum_{k\in\mathcal{E}} \left(f(\tau_k\omega) - f(\omega)\right). & \Delta^* = \Delta \le 0 \\ \text{multipl. ops.}: & M_k f(\omega) := v_k(\omega) f(\omega), & M_k^* = M_k \end{array}$$

and a commutation relation – due to (DFR):

$$\sum_{k\in\mathcal{E}} M_k \nabla_k + \sum_{k\in\mathcal{E}} \nabla_{-k} M_k = 0$$

The **infinitesimal generator** of the environment process:

$$L = \frac{1}{2}\Delta + \sum_{k\in\mathcal{E}} M_k \nabla_k = \frac{1}{2}\Delta - \sum_{k\in\mathcal{E}} \nabla_{-k} M_k =: -S + A$$

Formal solution of the eq. for gradient of harmonic coordinates:

$$\theta_k = |\Delta|^{-1/2} \nabla_k \Big(I + \underbrace{|\Delta|^{-1/2} A |\Delta|^{-1/2}}_{B} \Big)^{-1} |\Delta|^{-1/2} \varphi$$

Note:

$$\big\| |\Delta|^{-1/2} \nabla_k \big\| \leq 1, \qquad \big\| |\Delta|^{-1/2} \varphi \big\| < \infty, \qquad B \text{ unbuonded.}$$

Target: Prove that

$$B := \sum_{k \in \mathcal{E}} (|\Delta|^{-1/2} \nabla_{-k}) \, M_k \, |\Delta|^{-1/2}$$
$$B^* := |\Delta|^{-1/2} \sum_{k \in \mathcal{E}} M_k \, (\nabla_k |\Delta|^{-1/2}),$$

defined on appropriately chosen cores $\mathcal{B}, \mathcal{B}^* \subset \mathcal{L}^2(\Omega, \pi)$, make sense as *unbounded, skew-self-adjoint* operator.

Checking **von Neumann's criterion** $\texttt{Ker}(I \pm B^*) = \{0\}$:
boiles down to proving that there is no (non-trivial) solution to

$$\sum_{k \in \mathcal{E}} p_k(\omega)\theta_k(\omega) = 0,$$

$$\theta_k(\omega) + \theta_{-k}(\tau_k\omega) = 0$$

$$\theta_k(\omega) + \theta_l(\tau_k\omega) = \theta_l(\omega) + \theta_k(\tau_l\omega)$$

Gometrically: No non-trivial harmonic cocycles!

By-product: Uniqueness of the harmonic coordinates.

When (H_{-1}) fails: expect superdiffusive behaviour.

Diffusion in the curl of 2-dim Gaussian Free Field:

$v : \mathbb{R}^2 \mapsto \mathbb{R}^2, \quad v = \text{curl}\,(U * GFF), \quad dX(t) = dB(t) + v(X(t))dt.$

Theorem. [BT, B Valkó (*Journ. Stat. Phys.* 2012)]

$$d = 2: \quad t \log\log t \ll \mathbf{E}\left(|X(t)|^2\right) \ll t \log t, \quad (\texttt{expect} \asymp t\sqrt{\log t})$$

annealed, in the sense of Laplace transform (modulo Tauberian inversion).

Recent development:

G. Cannizzaro & L. Haunschmid-Sibitz & F. Toninelli (2021):

$$\mathbf{E}\left(|X(t)|^2\right) \asymp t\sqrt{\log t}$$

Go to Fabio's lecture on Friday.

RW on random Manhattan lattice:

Theorem. [S Ledger, BT, B Valkó (*ECP* 2018)]

$d = 2: \ t^{5/4} \ll \mathbf{E}\left(|X(t)|^2\right) \ll t^{3/2},$ (expect $\asymp t^{4/3}$, WIP)

$d = 3: \ t \log\log t \ll \mathbf{E}\left(|X(t)|^2\right) \ll t \log t,$ (expect $\asymp t\sqrt{\log t}$, WIP)

annealed, in the sense of Laplace transform (modulo Tauberian inversion).