An introduction to the representation theory of p-adic groups

Lecture 3: Bernstein decomposition and applications

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LMS-INI-Bath Summer School on
K-Theory & Representation Theory
23 July 2021

Recollection

Let ω be a fixed point for the action of W_Θ. We recall that ω is said to be good if for every point $\tau \in \mathcal{O}$, the decompositions $W_\tau = W'_\tau \rtimes R_\tau$ and $W_\omega = W'_\omega \rtimes R_\omega$ are compatible in the following sense :

1. we have $W'_\tau \subset W'_\omega$,
2. and the R-group R_τ is isomorphic with a subgroup of R_ω.

Example of a bad fixed point in $G = \mathrm{SL}_2(F)$

Let ω be the trivial character of the diagonal torus $T \simeq F^\times$. Then ω is a fixed point for the action of W_Θ and it is not good.

Proof

- We have $W_\Theta = W(G, T) = \mathbb{Z}/2$ and $R_\omega = \{1\}$.
- Let ϵ be the unique unramified quadratic character of $F^\times$ viewed as a character of T. Then, we have $R_{\omega\otimes\epsilon} = \mathbb{Z}/2\mathbb{Z}$. So our assumption definitely fails in that case. □

No good fixed-point !

Let $G = \mathrm{SL}_2(F)$ and ω the trivial character of T. For any $\chi \in \mathcal{X}_u(T)$, the representation $\omega \otimes \chi$ cannot at the same time be a fixed point for $W(G, T)$ and satisfy the conditions to be a good one. Indeed, assume that $\omega \otimes \chi$ is a fixed point for $W(G, T)$. The invariance under $W(G, T)$ implies that $\chi^2 = 1$. So we either have $\chi = 1$ or $\chi = \epsilon$. In either case, we have $R_{\sigma\otimes\chi} = \{1\}$. But the argument used above shows that by twisting with ϵ, we obtain a point with nontrivial R-group.

Notation

Let L be a Levi subgroup of a parabolic subgroup P of G.

- A character $\chi\colon L \to \mathbb{C}^\times$ is unramified if χ is trivial on every compact subgroup of L. We denote by $X_{\mathrm{nr}}(L)$ the group of unramified characters of L.
- Let σ be an irreducible supercuspidal smooth representation of L and $\mathcal{O}$ the set of equivalence classes of representations L of the form $\sigma \otimes \chi$, with $\chi \in X_{\mathrm{nr}}(L)$.
- We write $\mathfrak{s} := [L, \sigma]_G = (L, \mathcal{O})_G$ for the G-conjugacy class of the pair $(L, \mathcal{O})$ and $\mathfrak{B}(G)$ for the set of such classes $\mathfrak{s}$. We set $\mathfrak{s}_L := (L, \mathcal{O})_L$.

Bernstein decomposition

Let $\mathfrak{R}(G)$ be the category of all smooth complex representations of G.

- Let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every G-subquotient of π is equivalent to a subquotient of a parabolically induced representation $i_P^G(\sigma')$, where i_P^G is the functor of normalized parabolic induction and $\sigma' \in \mathcal{O}$.
- We denote by $\mathrm{Irr}^{\mathfrak{s}}(G)$ the set of irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(G)$.
- The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ in a direct product :

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

Remark

The Bernstein decomposition of the smooth dual of G says that

$$\mathrm{Irr}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G), \quad \text{where } \mathrm{Irr}^{\mathfrak{s}}(G) = \nu_{\mathfrak{B}}^{-1}(\mathfrak{s}).$$

We have

$$\mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}^{\mathfrak{t}}(G) = \prod_{\substack{\Theta \\ \mathfrak{z}(\Theta)\subset\mathfrak{s}}} \mathrm{Irr}^{\mathfrak{t}}_{\Theta}(G),$$

where $\mathfrak{z}\colon (M,\omega)_G \mapsto$ supercuspidal support of ω.

Notation

- Let $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$. We write $\mathfrak{s}_L = [L, \sigma]_L$.
- Let L^1 be the intersection of kernels of unramified characters of L.
- Let E be a carrier space for the supercuspidal representation σ of L.
- Let (σ_1, E_1) be an irreducible component of the restriction of σ to L^1.
- We denote by $\mathrm{c} - \mathrm{Ind}_{L^1}^{L}$ the functor of compact induction.

Theorem [Bernstein]

- The isomorphism class of

$$\Pi_L^{\mathfrak{s}_L} := \mathrm{c} - \mathrm{Ind}_{L^1}^{L}(\sigma_1, E_1)$$

  is independent of the choice of (σ_1, E_1).
- $\Pi_G^{\mathfrak{s}} := i_{L,P}^{G}(\Pi_G^{\mathfrak{s}_L})$ is a progenerator of $\mathfrak{R}^{\mathfrak{s}}(G)$.

Corollary

The functor $V \mapsto \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, V)$ is an equivalence from $\mathfrak{R}^{\mathfrak{s}}(G)$ to the category of modules of the algebra $\mathrm{End}_G(\Pi^{\mathfrak{s}})$:

$$\mathfrak{R}^{\mathfrak{s}}(G) \approx \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}.$$

We fix a Haar measure on G, write $\mathcal{H}(G)$ for the space of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$ and view $\mathcal{H}(G)$ as a $\mathbb{C}$-algebra via convolution relative to the Haar measure.

Let (ρ, V_ρ) be a smooth representation of a compact open subgroup K of G, and let $(\tilde\rho, V_{\tilde\rho})$ denote its contragredient. We define $\mathcal{H}(G,\rho)$ to be the space of compactly supported functions $f\colon G \to \mathrm{End}_G(V_{\tilde\rho})$ such that

$$f(kgk') = \tilde\rho(k)f(g)\tilde\rho(k'), \quad \text{where } k, k' \in K \text{ and } g \in G.$$

The convolution product gives $\mathcal{H}(G,\rho)$ the structure of a unitary associative $\mathbb{C}$-algebra.

Let $e_\rho \in \mathcal{H}(G)$ be the function defined by

$$e_\rho(g) := \begin{cases} \frac{\dim \rho}{\mathrm{meas}(K)} \mathrm{tr}(\rho(g^{-1})) & \text{if } g \in K, \\ 0 & \text{if } g \in G,\ g \notin K. \end{cases}$$

Then e_ρ is idempotent, and $e_\rho \star \mathcal{H}(G) \star e_\rho$ is a sub-algebra of $\mathcal{H}(G)$ with unit e_ρ.

Bushnell and Kutzko defined a canonical isomorphism :

$$\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V_\rho) \to e_\rho \star \mathcal{H}(G) \star e_\rho.$$

The algebras $\mathcal{H}(G, \rho)$ and $e_\rho \star \mathcal{H}(G) \star e_\rho$ are therefore canonically Morita equivalent. Hence, we get an equivalence of categories :

$$\mathcal{H}(G, \rho) - \mathrm{Mod} \approx e_\rho \star \mathcal{H}(G) \star e_\rho - \mathrm{Mod}.$$

Notation

We write $\mathfrak{R}_\rho(G)$ for the full sub-category of $\mathfrak{R}(G)$ whose objects are those V satisfying $V = \mathcal{H}(G) \star e_\rho \star V$, that is, $\mathfrak{R}_\rho(G)$ is generated over G by the subspace $e_\rho \star V$.

Definition

The pair (K, ρ) is called an $\mathfrak{s}$-type for G if the category $\mathfrak{R}_\rho(G)$ is closed by subquotients.

Example

The pair (I, triv), where I is an Iwahori subgroup of G is an $\mathfrak{i}$-type for $\mathfrak{i} = [T, \mathrm{triv}]_G$, where T is an F-split torus in G.

Proposition

The pair (K, ρ) is an $\mathfrak{s}$-type for G if and only if the following condition holds :

$$\pi_{|K} \text{ contains } \rho \leftrightarrow \nu_{\mathfrak{B}}(\pi) \in \mathfrak{s},$$

where $\nu_{\mathfrak{B}} : \mathrm{Irr}(G) \to \mathfrak{B}(G)$ is the inertial supercuspidal support map.

Definition

The pair (K, ρ) is said to be $\mathfrak{s}$-typical for G if the following condition holds :

$$\pi_{|K} \text{ contains } \rho \Rightarrow \nu_{\mathfrak{B}}(\pi) \in \mathfrak{s}.$$

Theorem [Bushnell-Kutzko]

If (K, ρ) is an $\mathfrak{s}$-type for G, then we have $\mathfrak{R}_\rho(G) = \mathfrak{R}^{\mathfrak{s}}(G)$ and the latter is equivalent to the category of modules of $\mathcal{H}(G, \rho)$:

$$\mathfrak{R}^{\mathfrak{s}}(G) \approx \mathcal{H}(G, \rho) - \mathrm{Mod}.$$

Notation

Let σ be a supercuspidal irreducible smooth representations of L. The group

$$X_{\mathrm{nr}}(L,\sigma) := \{\chi \in X_{\mathrm{nr}}(L) \,:\, \sigma \otimes \chi \cong \sigma\}$$

is finite.

Proposition

Recall that $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ denotes the set of irreducible objects of $\mathfrak{R}^{\mathfrak{s}_L}(L)$. There is a bijection

$$X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\sigma) \to \mathrm{Irr}^{\mathfrak{s}_L}(L) \,:\, \chi \mapsto \sigma \otimes \chi,$$

which endows $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ with the structure of a complex torus. Up to isomorphism this torus depends only on $\mathfrak{s}$, and it is known as the *Bernstein torus* $T^{\mathfrak{s}}$ attached to $\mathfrak{s}$. We note that $T^{\mathfrak{s}}$ is only an algebraic variety, it is not endowed with a natural multiplication map.

Let $\mathrm{Irr}(G)$ denote the set of irreducible objects of $\mathfrak{R}(G)$. The group $W(G, L) := \mathrm{N}_G(L)/L$ acts on $\mathrm{Irr}(L)$ by

$$w \cdot \pi = [\dot{w} \cdot \pi \;:\; l \mapsto \pi(w^{-1} l \bar{w})] \quad \text{for any lift } \dot{w} \in \mathrm{N}_G(L) \text{ of } w.$$

Bernstein also associated to $\mathfrak{s} = (L, \mathcal{X}(L) \cdot \sigma)_G$ the finite group

$$W^{\mathfrak{s}} := \{n \in \mathrm{N}_G(L) \;:\; {}^n\rho \sim \rho \otimes \chi \;\text{ for some } \chi \in \mathcal{X}(L)\} / L.$$

Remark

The group $W^{\mathfrak{s}}$ is an extended finite Weyl group that acts naturally on $T^{\mathfrak{s}}$, by automorphisms of algebraic varieties.

Proposition

The categorical centre of the Bernstein block $\mathfrak{R}^{\mathfrak{s}}(G)$ is

$$\mathrm{Z}(\mathfrak{R}^{\mathfrak{s}}(G)) \cong \mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}} = \mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}}).$$

Notation

Let Γ be a finite group acting as automorphisms of a complex affine variety X.

- For $x \in X$, let Γ_x denote the stabilizer group of x :

$$\Gamma_x := \{\gamma \in \Gamma \,:\, \gamma \cdot x = x\}.$$

- Let $\underline{\Gamma}_x$ be the set of conjugacy classes of Γ_x .
- Let $\mathrm{Irr}(\Gamma_x)$ be the set of equivalence classes of irreducible representations of Γ_x.
- We define

$$\widetilde{X}_{\mathrm{geo}} := \{(x, \underline{\gamma}) \in X \times \underline{\Gamma} \,:\, \underline{\gamma} \in \underline{\Gamma}_x\},$$

  and

$$\widetilde{X}_{\mathrm{spec}} := \{(x, \tau) \in X \times \mathrm{Irr}(\Gamma) \,:\, \tau \in \mathrm{Irr}(\Gamma_x)\}.$$

Γ acts

- on $\widetilde{X}_{\mathrm{geo}}$ by
$$\gamma' \cdot (x, \underline{\gamma}) := (\gamma' \cdot x, \gamma' \underline{\gamma} (\gamma')^{-1}),$$
- on $\widetilde{X}_{\mathrm{spec}}$ by
$$\gamma' \cdot (x, \tau) := (\gamma' \cdot x, \gamma'_* \tau),$$
where $\gamma'_* \colon \mathrm{Irr}(\Gamma_x) \to \mathrm{Irr}(\Gamma_{\gamma' x})$.

Definition

We define the *geometric* and *spectral extended quotients* of X by Γ as

$$X /\!/ \underline{\Gamma} := \widetilde{X}_{\mathrm{geo}} / \Gamma \quad \text{and} \quad X /\!/ \Gamma := \widetilde{X}_{\mathrm{spec}} / \Gamma,$$

respectively.

Remarks

- The quotient variety X/Γ is obtained by collapsing each orbit to a point, it is an affine variety. The geometric extended quotient is obtained by replacing the orbit of $x \in X$ by $\underline{\Gamma}_x$, while the spectral extended quotient is obtained by replacing it by the set $\mathrm{Irr}(\Gamma_x)$.
- For each $x \in X$, the sets $\underline{\Gamma}_x$ and $\mathrm{Irr}(\Gamma_x)$ are in bijection, but not in canonical way in general. It follows that the extended quotients $X/\!/\underline{\Gamma}$ and $X/\!/\Gamma$ are in bijection, but again not in canonical way in general.

The projections $(x, 0) \mapsto x$ and $(x, \tau) \mapsto x$ from $\widetilde{X}_{\mathrm{geo}}$ and $\widetilde{X}_{\mathrm{spec}}$ to X are Γ-equivariant and so pass to quotient spaces to give morphisms of affine varieties

$$\mathrm{pr}_{\mathrm{geo}} \colon X/\!/\underline{\Gamma} \to X/\Gamma \quad \text{and} \quad \mathrm{pr}_{\mathrm{spec}} \colon X/\!/\Gamma \to X/\Gamma.$$

Bernstein decomposition of $C_{\mathrm{r}}^*(G)$

We have

$$C_{\mathrm{r}}^*(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} C_{\mathrm{r}}^*(G)^{\mathfrak{s}}, \quad \text{where } C_{\mathrm{r}}^*(G)^{\mathfrak{s}} = \mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}^{\mathrm{t}}(G).$$

Notation

We set $T_{\mathrm{u}}^{\mathfrak{s}} := \mathcal{X}_{\mathrm{u}}(L) \cdot \sigma$, where σ is chosen to be unitary.

A new conjecture in the spirit of the BC-conjecture [A-Baum-Plymen (2011)]

We have

$$K_{W^{\mathfrak{s}}}^{j}(T_{\mathrm{u}}^{\mathfrak{s}}) \simeq K_j(C_{\mathrm{r}}^*(G)^{\mathfrak{s}}) \quad \text{for } j = 0, 1,$$

where $K_{W_{\mathfrak{s}}}^{j}(T_{\mathrm{u}}^{\mathfrak{s}})$ is the classical topological equivariant K-theory for the extended finite Weyl group $W^{\mathfrak{s}}$ acting on the compact torus $T_{\mathrm{u}}^{\mathfrak{s}}$.

Example

Suppose now that G is split, and consider the Iwahori-spherical point $\mathfrak{i} = [T, \mathrm{triv}]_G \in \mathfrak{B}(G)$, where T is maximal torus in G.
$W^{\mathfrak{i}}$ is the Weyl group W of G with respect to T and $T_{\mathrm{u}}^{\mathfrak{i}}$ is a maximal torus in the compact form of the Langlands dual group $G^\vee$.
The conjecture now takes the form

$$K_W^j(T_{\mathrm{u}}^{\mathfrak{i}}) \simeq K_j(C_{\mathrm{r}}^*(G)^{\mathfrak{i}}) \quad \text{for } j = 0, 1. \tag{1}$$

This statement is proved modulo torsion (A.-Baum-Plymen-Solleveld).
Applying the equivariant Chern character to the right-hand-side of (1) we obtain (modulo torsion) the total cohomology, even when $j = 0$, odd when $j = 1$, of the geometric extended quotient $T_{\mathrm{u}}^{\mathfrak{i}}/\!/\underline{W}$.

Some other directions

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Thank you very much for your attention !