An introduction to the representation theory of p-adic rgroups

Lecture 2: Tempered representations

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Definition

A representation (π, V) of a p-adic reductive group G is unitarizable if there exists a G-invariant (positive definite Hermitian) inner product on V.
Then, after completing V with respect to the given inner product, we obtain a unitary representation of G in a Hilbert space.

Proposition

The discrete series of G splits into two classes :

- supercuspidal representations : irreducible unitary representations whose matrix coefficients are compactly supported (mod the center of G),
- generalized special representations : irreducible unitary representations whose matrix coefficients are square-integrable (mod the center of G), and which are subrepresentations of representations induced from proper parabolic subgroups of G.

Example

The basic example of a discrete series representation that is (generally) not supercuspidal is the Steinberg representation.

Proposition

Suppose that (π, V) is irreducible smooth and square integrable modulo the center. Then (π, V) is unitarizable and there exists a unique $\mathrm{fdeg}(\pi) > 0$ such that

$$\int_{G/\mathrm{Z}_G} \langle u, \pi(g^{-1})u\rangle \cdot \langle v, \pi(g)v\rangle \mathrm{d}\bar{g} = \mathrm{fdeg}(\pi)^{-1}\langle \tilde{u}, v\rangle \cdot \langle \tilde{v}, u\rangle.$$

Definition

The constant $\mathrm{fdeg}(\pi)$ is called the formal degree of π. (Note that $\mathrm{fdeg}(\pi)$ depends on a choice of Haar measure on G/Z_G.)

Definition

If π is such that $\pi \otimes \chi$ is square integrable modulo the center for some quasi character (i.e., one-dimensional smooth representation) χ of G, then π is said to be essentially square integrable modulo the center, and we set

$$\mathrm{fdeg}(\pi) := \mathrm{fdeg}(\pi \otimes \chi).$$

Proposition

Let (τ, W) be an irreducible smooth representation of an open compact modulo center subgroup H of G. Suppose that the representation $\pi := \mathrm{c} - \mathrm{Ind}_H^G \tau$ is irreducible.
Then π is essentially square integrable modulo the center and

$$\mathrm{fdeg}(\pi) = \mathrm{meas}(H/\mathrm{Z}_G)^{-1} \deg(\tau).$$

The $\mathrm{SL}_2(F)$ case

The center of $\mathrm{SL}_2(F)$ is $\{\pm 1\}$. It follows that the discrete series of $\mathrm{SL}_2(F)$ consists of those irreducible unitary representations whose matrix coefficients lie in $L^2(\mathrm{SL}_2(F))$.

Proposition

Let $G = \mathrm{SL}_2(F)$ and χ be a character of the torus $T \simeq F^\times$.
Then the representation $\pi_\chi := i_{T,B}^G(\chi)$ is reducible if and only if χ is quadratic character of $F^\times$ (i.e., $\chi^2 = 1$ with $\chi \neq 1$), that is, for $\chi \in \{\mathrm{sgn}_{\varepsilon_F}, \mathrm{sgn}_{\varpi_F}, \mathrm{sgn}_{\varepsilon_F \varpi_F}\}$.
When π_χ is reducible, it is a direct sum of two irreducible components π_χ^+ and π_χ^-.

Classification of the irreducible unitary representations of $\mathrm{SL}_2(F)$ $(p \neq 2)$:

1. The unitary principal series π_χ, where $\chi \in \mathrm{Irr}(F^\times)$ and χ is not of order two. We observe that $\pi_\chi \simeq \pi_{\chi^{-1}}$;
2. The irreducible components π_χ^+ and π_χ^- of the principal series π_χ, where χ is a quadratic character of $F^\times$.
3. The supercuspidal discrete series.
4. The complementary series $\pi_\chi = i_{T,B}^G(\chi_s)$, where $\chi_s(t) := |t|_p^s$ and $0 < s < 1$.
5. Two additional irreducible unitary representations : these are obtained by considering the representation π_χ, where $\chi(x) = |x|$, which occurs at the "end of the complementary series". The representation π_χ has a composition series of length 2 : one subquotient is the trivial representation, and the other is the Steinberg representation, which is square-integrable.

Characteristic spaces

Let $P = LU$ be a parabolic subgroup of G, (π, V) an admissible representation of G, χ a smooth character of A_L (the F-points of $\mathbf{A_L}$, the maximal F-split torus lying in the center of $\mathbf{L}$), K_L an open compact subgroup of L. We define $(r^G_{L,P}V)^{K_L}_\chi$ to be the set of $v \in (r^G_{L,P}V)^{K_L}$ such that there exists $d \in \mathbb{Z}_{>0}$ with

$$\left(r^G_{L,P}(\pi)(t) - \chi(t)\mathrm{Id}_V\right)^d \cdot v = 0 \quad \text{for all } t \in A,$$

We have

$$(r^G_{L,P}V)^{K_L} = \bigoplus_\chi (r^G_{L,P}V)^{K_L}_\chi,$$

where χ describes the set of smooth characters of A.

Exponents

If $K_L' \subset K_L$ is another open compact subgroup of G, we have $(r^G_{L,P}V)^{K_L}_\chi \subset (r^G_{L,P}V)^{K_L'}_\chi$, and we set

$$(r^G_{L,P}V)_\chi := \bigcup_{K_L} (r^G_{L,P}V)^{K_L}_\chi,$$

where K_L runs over the set of open compact subgroups of G.

Definition

If $(r^G_{L,P}V)_\chi \neq \{0\}$, we say that χ is an exponent of π for P.
The set of these exponents will be denoted by $\mathrm{Exp}(A_L, r^G_{L,P}V)$.

Notation

Fix a minimal parabolic subgroup $P_0 = L_0U_0 \subset P$ such that $A_L \subset A_0 := A_{L_0}$. Let $X^*(A_L)$ be the group of rational characters of A_L, and $X_*(A_L) := \mathrm{Hom}_{\mathbb{Z}}(X_*(A_L), \mathbb{Z})$. We write

$$\mathfrak{a}_L := X_*(A_L) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \mathfrak{a}_L^* := X^*(A_L) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\Sigma(A_0)$ be the set of roots of A_0 in $\mathfrak{g}$, and $\Sigma_{\mathrm{red}}(A_0)$ the subset of reduced roots : $\Sigma_{\mathrm{red}}(A_0) := \{\alpha \in \Sigma(A_0) \,:\, \alpha/m \notin \Sigma(A_0) \text{ if } m \geq 2\}$. Let Δ_0 be the set of simple roots in $\Sigma_{\mathrm{red}}(A_0)$, and set

$$\Delta(P) := \{\alpha_{|\mathfrak{a}_L} \,:\, \alpha \in \Delta_0 \backslash \Delta_0^L\},$$

where Δ_0^L is the analog of Δ_0 where G is replaced by L.

Casselman's criterion

Let $(\pi, V) \in \mathrm{Irr}(G)$ with unitary central character. Then π is square integrable modulo the center if and only if, for any parabolic subgroup $P = LU$ of G and any $\chi \in \mathrm{Exp}(A, r^G_{L,P} V)$, we have $\Re(\chi) \in {}^+[\mathfrak{a}_L^*]^G_P$, where

$$^+[\mathfrak{a}_L^*]^G_P := \{ \sum_{\alpha \in \Delta(P)} m_\alpha \alpha \, : \, m_\alpha > 0 \}.$$

Notation

We write

$$^+\overline{[\mathfrak{a}_L^*]}^G_P := \{ \sum_{\alpha \in \Delta(P)} m_\alpha \alpha \, : \, m_\alpha \geq 0 \}.$$

Definition

A representation (π, V) of G is tempered if it is admissible, and if for any parabolic subgroup $P = LU$ of G, every $\chi \in \mathrm{Exp}(A, r^G_{L,P} V)$ satisfies $\Re(\chi) \in {}^+\overline{[\mathfrak{a}_L^*]}^G_P$.

Proposition

An irreducible unitary representation of G is tempered if and only if its matrix coefficients lie in $L^{2+\epsilon}(G/Z)$ for all $\epsilon > 0$.

Notation

Let $\mathrm{Irr}^2(G)$ denote the set of equivalence classes of irreducible square integrable modulo the center representations of G and let

$$\mathrm{Irr}^{\mathrm{t}}(G) \supset \mathrm{Irr}^2(G)$$

be the set of equivalence classes of irreducible tempered representations of G.

Theorem [Harish-Chandra]

Any irreducible tempered representation π of G occurs as an irreducible component of a parabolically induced representation $i_{M,Q}^G(\omega)$, where Q is a parabolic subgroup of G with Levi factor M and $\omega \in \mathrm{Irr}^2(M)$.
The G-conjugacy class $(M,\omega)_G$ of (M,ω) is uniquely determined and is called the discrete support of π.

Notation

We write

$$\Omega(G) := \{(L,\sigma)_G \,:\, L \text{ Levi subgroup of } G,\ \sigma \text{ irred. supercuspidal}\}$$

$$\Omega^{\mathrm{t}}(G) := \{(M,\omega)_G \,:\, M \text{ Levi subgroup of } G,\ \omega \text{ discrete series}\}$$

and denote by

$$\nu\colon \mathrm{Irr}^{\mathrm{t}}(G) \to \Omega(G) \quad \text{and} \quad \nu^{\mathrm{t}}\colon \mathrm{Irr}^{\mathrm{t}}(G) \to \Omega^{\mathrm{t}}(G)$$

the corresponding supercuspidal support map and discrete support map.

Compatibility of the discrete support and supercuspidal maps

The following diagram commutes :

$$\begin{array}{ccccc} & & \mathrm{Irr}^{\mathrm{t}}(G) & & \\ & \swarrow^{\nu^{\mathrm{t}}} & & \searrow^{\nu} & \\ \Omega^{\mathrm{t}}(G) & & \xrightarrow{\quad \mathfrak{z} \quad} & & \Omega(G) \end{array}$$

where $\mathfrak{z}\colon (M,\omega)_G \mapsto$ supercuspidal support of ω.

Remark

For $G = \mathrm{GL}_n(F)$, the map $\mathfrak{z}$ is injective and ν^{t} is bijective (since unitary induction is irreducible for $\mathrm{GL}_n(F)$).

Unitary unramified characters

Definition : A character $\chi\colon M \to \mathbb{C}^\times$ is unramified if χ is trivial on every compact subgroup of M.
Notation : Let $\mathcal{X}(M)$ be the abelian group of unramified characters of M, and $\mathcal{X}_{\mathrm{u}}(M)$ the subgroup of the unitary unramified characters.

Orbits under unitary unramified characters

Let ω be a square-integrable irred. repres. of M.
Let $\mathcal{O}$ denote the orbit of ω under $\mathcal{X}_{\mathrm{u}}(M)$:

$$\mathcal{O} := \{\omega \otimes \chi \,:\, \chi \in \mathcal{X}_{\mathrm{u}}(M)\} = \mathcal{X}_{\mathrm{u}}(M) \cdot \omega.$$

Notation/Definition

- Let $\mathcal{P}$ be the set of pairs $(Q = MN, \mathcal{O})$, where Q is a semi-standard parabolic subgroup of G and $\mathcal{O}$ as above.
- Two pairs $(Q = MN, \mathcal{O})$ and $(Q' = M'N', \mathcal{O}')$ in $\mathcal{P}$ are associated if there exists $w \in W^G$ such that $w \cdot M = M'$ and $w\mathcal{O} = \mathcal{O}'$.
- We fix a set $\tilde{\mathcal{P}}$ of representatives in $\mathcal{P}$ for the classes of association.
- We write $W(M, \mathcal{O}) := \{n \in \mathrm{N}_G(M) \, : \, w(\mathcal{O}) = \mathcal{O}\} \,/M$.
- For each $f \in C_c^\infty(G)$, the function $f^\vee$ is defined by

$$f^\vee(g) := f(g^{-1}) \quad \text{for } g \in G.$$

- Recall : if (π, V) is a smooth representation of G, given $f \in C_c^\infty(G)$, the operator $\pi(f)\colon V \to V$ is defined by

$$\pi(f)v := \int_G f(g)\pi(g)v\mathrm{d}g, \quad \text{for } v \in V.$$

The Plancherel formula after Harish-Chandra [Waldspurger, JIMJ 2003]

For each $f \in C_c^\infty(G)$ and each $g \in G$, we have

$$f(g) = \sum_{(Q,\mathcal{O})\in\tilde{\mathcal{P}}} c_{G,M}|W(M,\mathcal{O})|^{-1} \int_{\mathcal{O}} \mu_{G|M}(\omega)\cdot \mathrm{fdeg}(\omega)\cdot \theta_\omega^G(\lambda(g)f^\vee)\,\mathrm{d}\omega,$$

where

$$\theta_\omega^G := \mathrm{tr}(\pi(f)), \quad \text{where } \pi = i_{M,Q}^G(\omega),$$

$c_{G,M}$ is a constant depending on G and M, and $\mu_{G|M}$ is the Plancherel measure defined by Harish-Chandra.

Definition

The G-conjugacy class $\Theta := (M, \mathcal{O})_G$ of a discrete pair $(M, \mathcal{O})$ is called an inertial discrete pair in G. We write also $\Theta = [M, \omega]_G$ for $\omega \in \mathcal{O}$. We denote by $\mathfrak{B}^2(G)$ the set of inertial discrete pairs in G.

Notation

If Q is a parabolic subgroup of G with Levi factor M, then $\mathrm{Irr}^{\mathrm{t}}_{\Theta}(G)$ is defined to be the set of irreducible representations of G that occur in one of the induced representations $i^G_{M,Q}(\omega \otimes \chi)$, $\chi \in \mathcal{X}_{\mathrm{u}}(M)$:

$$\mathrm{Irr}^{\mathrm{t}}_{\Theta}(G) := (\nu^{\mathrm{t}})^{-1}(\Theta).$$

A decomposition of Bernstein type of $\mathrm{Irr}^{\mathrm{t}}(G)$ [Plymen 1990]

$$\mathrm{Irr}^{\mathrm{t}}(G) = \bigsqcup_{\Theta \in \mathfrak{B}^2(G)} \mathrm{Irr}^{\mathrm{t}}_{\Theta}(G).$$

Goal : to describe the structure of the $\mathrm{Irr}^{\mathrm{t}}_{\Theta}(G)$'s up to strong Morita equivalence.

Definition

Choose a left-invariant Haar measure on G and form the Hilbert space $L^2(G)$.
The left regular representation λ of $L^1(G)$ is given by $(\lambda(f))(h) := f * h$, where $f \in L^1(G)$ and $h \in L^2(G)$, and $*$ denotes convolution.

Definition

The reduced C^*-algebra of G, denoted as $C^*_{\mathrm{r}}(G)$, is the C^*-algebra generated by the image of λ.
Its spectrum may be identified to the tempered dual $\mathrm{Irr}^{\mathrm{t}}(G)$ of G.

The C^*-blocks decomposition

We will described a decomposition :

$$C_{\mathrm{r}}^*(G) = \bigoplus_{\Theta \in \mathfrak{B}^2(G)} C_{\mathrm{r}}^*(G; \Theta),$$

where $C_{\mathrm{r}}^*(G;\Theta)$ is a subalgebra of $C_{\mathrm{r}}^*(G)$ with spectrum $\mathrm{Irr}_{\Theta}^{\mathrm{t}}(G)$.

Let $\omega \in \mathrm{Irr}^2(M)$. We fix a Haar measure on G and a maximal compact subgroup K of G, with measure of total mass one. We may and will assume that K is a good special maximal compact subgroup in "good relative position" with M (in particular, $G = KQ$).

Notation

- $E_Q(\tau)$, for $(\tau, V_\tau) \in \mathcal{O}$, is the space of functions $f\colon G \to V_\tau$ that are right-invariant under some compact open subgroup of G and satisfy

$$f(xmn) = \delta_Q^{-\frac{1}{2}}(m)(\tau)(m)^{-1}f(x), \quad \text{for } x \in G,\ m \in M,\ n \in N.$$

  and $\mathrm{I}_Q(\tau)$ the representation of G on $E_Q(\tau)$ by left translation.
- E_Q^K the space of functions $f\colon K \to V_\tau$ that are right-invariant under some open subgroup of K and satisfy

$$f(kmn) = \tau(m)^{-1}f(k), \quad \text{for } k \in K,\ m \in M \cap K,\ n \in N \cap K.$$

Let $\mathcal{H}_Q$, $\mathcal{H}_Q^K$ denote the Hilbert completions of $E_Q(\tau)$, E_Q^K with respect to the inner product

$$\langle f, f' \rangle = \int_K \langle f(k), f'(k) \rangle_{V_\tau} dk.$$

We still write $\mathrm{I}_Q(\tau)$ for the representation of G on $\mathcal{H}_Q(\tau)$.

For $\chi \in \mathcal{X}_{\mathrm{u}}(M)$, we consider the linear isomorphism

$$F_Q^K(\chi): \quad \begin{array}{ccc} \mathcal{H}_Q(\omega \otimes \chi) & \to & \mathcal{H}_Q^K \\ f & \mapsto & f|_K \end{array}.$$

- Then $\mathrm{I}_Q^K(\omega \otimes \chi)$ denotes the representation of G on $\mathcal{H}_Q^K$ such that

$$\mathrm{I}_Q^K(\omega \otimes \chi)(g) = F_Q^K(\chi) \circ \mathrm{I}_Q(\omega \otimes \chi)(g) \circ F_Q^K(\chi)^{-1}.$$

- For every smooth and compactly supported function $f \in \mathcal{C}_c^\infty(G)$, the operator

$$\pi_\tau(f) := \int_G f(g)\mathrm{I}_Q^K(\tau)(g)dg.$$

  It is compact, and depends continuously on τ.
- Let $\mathfrak{K}(\mathcal{H}_Q^K)$ be the algebra of compact operators on $\mathcal{H}_Q^K$.

Definition of the subalgebra $C_r^*(G;\Theta)$

- We obtain a linear map

$$\begin{array}{ccc} C_c^\infty(G) & \to & \mathcal{C}(\mathcal{O}, \mathfrak{K}(\mathcal{H}_Q^K)) \\ f & \mapsto & (\tau \mapsto \pi_\tau(f)). \end{array}$$

- Upon completing $C_c^\infty(G)$ to $C_r^*(G)$, there arises a C^*-morphism

$$C_r^*(G) \to \mathcal{C}(\mathcal{O}, \mathfrak{K}(\mathcal{H}_Q^K)).$$

Let $C_r^*(G;\Theta)$ denote the image of $C_r^*(G)$ by this morphism.

Conclusion

From the above discussion, we obtain a morphism

$$C_r^*(G) \to \bigoplus_{\Theta \in \mathfrak{B}^2(G)} C_r^*(G;\Theta). \qquad (1)$$

Theorem [Plymen 1990]

The map in (1) is a C^*-isomorphism.

The stabilizer of $\mathcal{O}$

Let $\Theta = [M, \omega]_G \in \mathfrak{B}^2(G)$. We set

$$\mathrm{N}_G(\Theta) := \{n \in \mathrm{N}_G(M) \,:\, {}^n\omega \simeq \omega \otimes \chi \text{ for some } \chi \in \mathcal{X}_{\mathrm{u}}(M)\},$$

and define the stabilizer of $\mathcal{O}$ (so called the inertial stabilizer of ω) as :

$$W_\Theta := \mathrm{N}_G(\Theta)/M.$$

Structure of $C_r^*(G;\Theta)$ [Afgoustidis-A.]

When $F = \mathbb{R}$, Wassermann gave a simple determination of $C_r^*(G;\Theta)$ up to strong Morita equivalence. We will see that an analogous results holds when F is p-adic, under the assumption that the action of W_Θ on $\mathcal{O}$ admits a good fixed-point.

- For $\omega \in \mathrm{Irr}^2(M)$, we denote by W_ω its stabilizer in $\mathrm{N}_G(M)/M$:

$$W_\omega := \{n \in \mathrm{N}_G(M) \mid \ {}^n\omega \simeq \omega\ \}/M,$$

  where ${}^n\omega\colon m \mapsto \omega(n^{-1}mn)$ of M.

- **Knapp-Stein decomposition** of W_ω : we have $W_\omega = W'_\omega \rtimes R_\omega$, where :
  - **Definition of W'_ω :** To each $\alpha \in \Delta$ is attached a Levi subgroup $M_\alpha \supset M$. Let Δ be the set of roots for (G, M). We have $\mu^{M_\alpha} : \mathcal{O} \to \mathbb{R}^+$. The set

$$\Delta' := \left\{\alpha \in \Delta\ :\ \mu^{M_\alpha}(\omega) = 0\right\}$$

    is itself a root system, and the group W'_ω is its Weyl group.
  - **Definition of R_ω :** We fix a positive system Δ'_+ in Δ', and set

$$R_\omega := \left\{w \in W_\omega\ :\ w(\Delta'_+) = \Delta'_+\right\}.$$

Remark

Obviously : $W_\omega \subset W_\Theta$ for any $\omega \in \mathcal{O}$.

Proposition

When $F = \mathbb{R}$, the action of W_Θ on $\mathcal{O}$ has always a fixed point, i.e., there exists $\omega \in \mathcal{O}$ such that $W_\omega = W_\Theta$.

Question (open in general)

When F is p-adic and $\Theta = (M, \mathcal{O})_G \in \mathfrak{B}^2(G)$, does the action of W_Θ on $\mathcal{O}$ always admit a fixed point ?

Theorem [Afgoustidis-A. 2020]

Let G be a quasi-split symplectic, orthogonal or unitary group over a p-adic field F. Then for every $\Theta = (M, \mathcal{O})_G \in \mathfrak{B}^2(G)$, the action of W_Θ on $\mathcal{O}$ has a fixed point.

Assumption

We fix $\Theta \in \mathfrak{B}^2(G)$ and we assume that the action of W_Θ on $\mathcal{O}$ has a fixed-point, say ω.

Definition

The fixed point ω is good if for every point $\tau \in \mathcal{O}$, the Knapp-Stein decompositions $W_\tau = W'_\tau \rtimes R_\tau$ and $W_\omega = W'_\omega \rtimes R_\omega$ are compatible in the following sense :

1. we have $W'_\tau \subset W'_\omega$,
2. and the R-group R_τ is isomorphic with a subgroup of R_ω.

Intertwining operators (works of Waldspurger, Langlands, Arthur)

For each $w \in W_\Theta$ and every $\chi \in \mathcal{X}_{\mathrm{u}}(M)$, there is an operator

$$\mathcal{A}(w, \omega \otimes \chi) \colon \mathcal{H}_Q^K \to \mathcal{H}_Q^K$$

that intertwines $\mathrm{I}_Q^K(\omega \otimes \chi)$ and $\mathrm{I}_Q^K(\omega \otimes (w\chi))$, and satisfies

$$\mathcal{A}(w_1 w_2, \omega \otimes \chi) = \eta_\omega(w_1, w_2)\, \mathcal{A}(w_1, \omega \otimes (w_2\chi))\, \mathcal{A}(w_2, \omega \otimes \chi),$$

for any w_1, w_2 in W_Θ and $\chi \in \mathcal{X}_{\mathrm{u}}(M)$, where

$$\eta_\omega \colon W_\Theta \times W_\Theta \to \mathbb{C}^\times \text{ is a 2-cocycle.}$$

Proposition

For every $\tau \in \mathcal{O}$, the map $r \mapsto \mathcal{A}(r, \tau)$ defines a projective representation of R_τ on $\mathcal{H}_Q^K$. The multiplier of this projective representation is the restriction $\eta_\tau \colon R_\tau \times R_\tau \to \mathbb{C}^\times$ of the cocycle η_ω.

Central extensions of R-groups [Arthur]

The theory of the R-group can be brought to full fruition once chosen a central extension

$$1 \to Z_\omega \to \widetilde{R}_\omega \to R_\omega \to 1$$

with the property that the 2-cocycle of $\widetilde{R}_\omega$ induced by η_ω is a coboundary.

Theorem [Arthur]

The irreducible components of $i_{M,Q}^G(\omega)$ are in natural bijection with the set of irreducible representations of $\widetilde{R}_\omega$ with Z_ω-central character ζ_ω, where $\zeta_\omega \colon Z_\omega \to \mathbb{C}^\star$ is given by $\zeta_\omega(z) := \xi_\omega(z)^{-1}$ for $z \in Z_\omega$.

The extension $\widetilde{R}_\omega$ comes with a central idempotent $\widetilde{p}$ acting on the group algebra of $\widetilde{R}_\omega$: it is a minimal idempotent in $\mathbb{C}[Z_\omega]$ such that

$$\widetilde{p}\,\mathbb{C}[\widetilde{R}_\omega] \simeq \mathbb{C}[R_\omega, \eta_\omega].$$

Theorem [Afgoustidis-A. 2020]

Assume that ω is a good fixed-point for the action of W_Θ on $\mathcal{O}$. Then we have the strong Morita equivalence

$$C_r^*(G,\Theta) \underset{\text{Morita}}{\sim} \widetilde{p}\left[\, \mathcal{C}(\mathcal{O}/W'_\omega) \rtimes \widetilde{R}_\omega \,\right].$$

How to interpret the result above :

- Strong Morita equivalence preserves spectra : if A and B are strongly Morita equivalent C^*-algebras, then their spectra are homeomorphic.
- We have a strong Morita equivalence between a highly noncommutative C^*-block and (the image under $\tilde{p}$ of) an almost commutative C^*-algebra, namely the crossed product of a commutative C^*-algebra by a finite group. This allows one to infer the topology on the tempered dual, in a way which reflects reducibility of the induced representations.