An introduction to the representation theory of p-adic groups

Lecture 1: Basic notions on p-adic reductive groups and their representations

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When we complete the rational numbers $\mathbb{Q}$ with respect to the usual absolute value, we obtain the field of real numbers $\mathbb{R}$. Let p be a prime number. If we complete $\mathbb{Q}$ with respect to the so-called p-adic absolute value, we get the field $\mathbb{Q}_p$ of p-adic numbers.

We denote by $\mathrm{ord}_p(n)$ the exponent of p in the factorization of an integer n as product of prime numbers. Then the p-adic absolute value $|\ |_p \colon \mathbb{Q} \to \mathbb{R}^\times$ is defined by $|0|_p := 0$ and

$$\left|\frac{a}{b}\right|_p := p^{\mathrm{ord}_p(b)-\mathrm{ord}_p(a)} \quad \text{when } a \text{ and } b \text{ are non-zero integers.}$$

The ring of integers of $\mathbb{Q}_p$ is the ring of p-adic integers, defined as

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \,:\, |x|_p \leq 1\}\,.$$

p-adic fields

Let F be a finite extension of $\mathbb{Q}_p$. There is a unique absolute value $|\ |_F$ on F extending $|\ |_p$. It satisfies $|x+y|_F \leq \max(|x|_F, |y|_F)$ for any $x, y \in F$.

- The ring of integers of F is $\mathfrak{o}_F := \{x \in F : |x|_F \leq 1\}$, it is a local ring with maximal ideal $\mathfrak{p}_F = \{x \in F : |x|_F < 1\}$. A generator ϖ_F of $\mathfrak{p}_F$ is called a uniformizer of F.
- The residual field $k_F := \mathfrak{o}_F/\mathfrak{p}_F$ of F is a finite field of characteristic p (the residual characteristic of F) and of cardinality denoted by q (a power of p).
- The units in $\mathfrak{o}_F$ form a group $\mathfrak{o}_F^\times = \{x \in F : |x|_F = 1\}$.
- We have $\mathfrak{o}_F \simeq \varprojlim \mathfrak{o}_F/\mathfrak{p}_F^m$ $(m \geq 1)$.

Definition

A *p-adic reductive group* G is the group of F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F, where F is a finite extension of $\mathbb{Q}_p$ with p a prime number.

Definition

A topological group G is profinite if it is compact and totally disconnected (that is, any connected subset of G is a singleton).

Example

A finite group, with the discrete topology, is profinite.

Characterization

If G is profinite then there is a topological isomorphism $G \simeq \varprojlim G/N$, where N runs through all open, normal subgroups of G, hence G is the projective limit of a system of finite groups.
Conversely such a projective limit is a compact and totally disconnected group, hence profinite.

Definition

A topological group G is locally profinite if it satisfies one of the equivalent following conditions :

1. G is locally compact and totally disconnected ;
2. every neighbourhood of the identity in G contains a compact open subgroup.

Remark

Compact locally profinite groups G are profinite.

Example 1

The additive group of $\mathbb{Q}_p$ is a locally profinite group, with a fundamental system of neighbourhoods of 0 given by $p^m\mathbb{Z}_p$, $m \in \mathbb{Z}$. The same indeed holds for the additive group of a p-adic field F and the $\mathfrak{p}_F^m$, $m \in \mathbb{Z}$, constitute a family of compact open subgroups such that $F = \bigcup_{m\in\mathbb{Z}} \mathfrak{p}_F^m$.

Example 2

The multiplicative group $F^\times$ is locally profinite with a fundamental system of neighbourhoods of 1 given by the compact open subgroups $1 + \mathfrak{p}_F^m$, $m \geq 1$, contained in the unique maximal compact subgroup $\mathfrak{o}_F^\times$.

Definition

A representation (π, V) of a locally profinite group G is

- smooth if for any $v \in V$, its stabilizer $G_v := \{g \in G : \pi(g)(v) = v\}$ in G is an open subgroup of G;
- admissible if for every compact open subgroup K of G, the dimension of V^K is finite.

Theorem

Every irreducible smooth representation of G is admissible.

Reductive groups

Let F be a p-adic field and $\mathbf{G}$ a connected reductive algebraic group defined over F. We denote by $G = \mathbf{G}(F)$ the group of F-points of $\mathbf{G}$. Then G is called a p-adic reductive group.

Examples of split p-adic reductive groups :

- $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$,
- $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_n(F)$ (classical groups),
- groups of exceptional type G_2, F_4, E_6, E_7 and E_8.

Remark

The group $\mathbf{G}(\mathfrak{o}_F)$ is a maximal open compact subgroup of G.

Proposition

If (π, V) is a representation of G, then (π, V^∞) is a smooth subrepresentation of V, where $V^\infty := \{v \in V \,:\, G_v \text{ is open}\}$.

Notation

Let $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. If $\lambda \in V^*$ and $v \in V$, let $\langle \lambda, v\rangle$ denote the value of the linear functional λ on the vector v, and define $\pi^*(g)\lambda \in V^*$ by

$$\langle \pi^*(g)\lambda, v\rangle := \langle \lambda, \pi(g)^{-1}v\rangle, \quad \text{for } v \in V.$$

Definition

(π^*, V^*) is a representation of V.

Remark

Even if π is smooth, the representation π^* may not be smooth.

Definition

Let $\tilde{\pi}$ be the subrepresentation $\tilde{V} := V^{*,\infty}$ of π^*. This representation is called the contragredient of (π, V).

Remark

We have $\tilde{\tilde{\pi}} \simeq \pi$, if π is admissible.

Proposition/Definition

If π is an irreducible smooth representation of G, then the center of G acts by a character. This is called the central character of π.

Remark

Smoothness is preserved by surjective morphisms and by the operation of taking subrepresentations, subquotients, direct sums.

Definition of smooth induction

Let H be a closed subgroup of G and (τ, W) a smooth representation of H. The space $\mathrm{Ind}_H^G(W)$ of functions $f\colon G \to W$ such that

- for all $h \in H$ and $g \in G$, we have $f(hg) = \tau(h)f(g)$,
- there exists an open subgroup K_f of G such that for all $k \in K_f$ and $g \in G$, we have $f(gk) = f(g)$,

is stable under right translations by elements of G and the representation $(\mathrm{Ind}_H^G\tau, \mathrm{Ind}_H^G W)$ of G is the representation (smoothly) induced by τ.

Definition of compact induction

The subspace $\mathrm{c}-\mathrm{Ind}_H^G(W)$ of $\mathrm{Ind}_H^G(W)$ formed by functions with compact support modulo H (that is, the support is contained in some $H\Omega$ where Ω is a compact set in G) is G-stable and provides a subrepresentation $(\mathrm{c}-\mathrm{Ind}_H^G(\tau), \mathrm{c}-\mathrm{Ind}_H^G(W))$ of G called the representation compactly induced by τ.
The two representations $\mathrm{c}-\mathrm{Ind}_H^G(W)$ and $\mathrm{Ind}_H^G(W)$ coincide when the quotient space G/H is compact.

In order to simply the exposition, we will assume that G is quasi-split, that is, that there is a Borel subgroup of $\mathbf{G}$ defined over F.

Definitions

- A parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is an algebraic subgroup which contains a Borel subgroup. Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{P}$.
- If $\mathbf{P}$ is defined over F, we will call it an F-parabolic subgroup.
- A Levi subgroup $\mathbf{L}$ of an F-parabolic subgroup $\mathbf{P}$ is a reductive group defined over F such that the mapping $(l, u) \mapsto lu$ defines an F-isomorphism of the algebraic varieties $\mathbf{L} \times \mathbf{U}$ and $\mathbf{P}$. Such a subgroup $\mathbf{L}$ always exists and is connected.
- Let $\mathbf{A} = \mathbf{A_L}$ be a maximal F-split torus lying in the center of $\mathbf{L}$. Then $\mathbf{A}$ is unique and $\mathbf{L}$ is the centralizer of $\mathbf{A}$ in $\mathbf{G}$. We call $\mathbf{A}$ the split component of $\mathbf{L}$.
- The group $P := \mathbf{P}(F)$ is called a parabolic subgroup of G with unipotent radical $U := \mathbf{U}(F)$ and Levi factor $L := \mathbf{L}(F)$.

We have $P = LU$ and there is a canonical isomorphism $L \simeq P/U$ obtained by composing the injection $L \hookrightarrow P$ with quotient map $P \to P/U$.
Then P is a closed subgroup of G with a compact quotient G/P and we can induce representations from P to G. In this case there is no difference between smooth induction and compact induction.

Example 1

Let $G = \mathrm{GL}_n(F)$ be the group of invertible n by n matrices with coefficients in F.

- The group $\mathrm{GL}_n(\mathfrak{o}_F)$ is a maximal compact subgroup of G, and any maximal compact subgroup of G is conjugate to $\mathrm{GL}_n(\mathfrak{o}_F)$.
- Up to conjugacy, a parabolic subgroup P is a subgroup of upper block-triangular matrices, U is formed by those matrices whose diagonal blocks are identity matrices, and L is the subgroup of block-diagonal matrices, a product of smaller $\mathrm{GL}_{n_i}(F)$'s with $n_1 + \cdots + n_r = n$.

Example 2

Let $G := \mathrm{SL}_2(F)$ be the group of 2 by 2 matrices with coefficients in F and determinant equal to 1. The group $\mathrm{SL}_2(\mathfrak{o}_F)$ is a maximal compact subgroup of $\mathrm{SL}_2(F)$ with volume $\frac{q^2-1}{q^2}$. We set

$$T := \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in F^\times \right\}.$$

The subgroup T is topologically isomorphic to $F^\times$. We set

$$U := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in F^\times, b \in F \right\} \quad \text{and} \quad \overline{U} := \left\{ \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right) \mid a \in F^\times, c \in F \right\}.$$

The subgroups U and $\overline{U}$ are topologically isomorphic to F. The subgroup B is called a Borel subgroup of G, and B is a semidirect product of T and U, with U the normal subgroup. If χ is any character of T, we may extend it in a trivial way to a character χ_B of B by setting $\chi(tu) := \chi(t)$ for any $t \in T$ and $u \in U$. The normality of U implies that the extended character is well-defined.

Notation

Let $\mathfrak{R}(G)$ denote the category of smooth representations of G, and similarly for P and L.

Definition

The inflation of a representation σ of L to P is the unique representation of P which restricts to σ on L and trivial on U; we denote it as $\mathrm{infl}_L^P(\sigma)$.

Definition

The parabolic induction functor is the following composition

$$i_{L,P}^G \colon \mathfrak{R}(L) \xrightarrow{\mathsf{inf}_L^P} \mathfrak{R}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}(G).$$

U-coinvariant

Since U is normal in P, for any smooth representation (τ, V) of P the space

$$V(U) := \langle \tau(u)v - v \,:\, u \in U, v \in V \rangle$$

is stable under P and the quotient $V/V(U)$ provides a smooth representation of P that is trivial on U, i.e., a smooth representation of L.

Definition

The left adjoint functor to parabolic induction is obtained by composing restriction with the U-coinvariants functor, which is left adjoint to inflation. The parabolic restriction functor, or Jacquet (restriction) functor, is the following composition

$$r_{L,P}^{G} \colon \mathfrak{R}(G) \xrightarrow{\mathrm{Res}_P^G} \mathfrak{R}(P) \xrightarrow{U\text{-coinvariants}} \mathfrak{R}(L).$$

Proposition

Let $P = LU$ be a parabolic subgroup of G. Then the functors $i_{L,P}^G$ and $r_{L,P}^G$ are both exact, and $r_{L,P}^G$ is the left adjoint of $i_{L,P}^G$.

Theorem [Bernstein]

The right adjoint of $i_{L,P}^G$ is the functor $r_{L,\overline{P}}^G$, where $\overline{P}$ denotes the opposite parabolic subgroup of P.

Definition

The set of all irreducible representations of G which are subquotients of $i_{T,B}^G(\chi)$, with B a Borel subgroup of G and χ of character of a torus $T \subset B$ is called the principal series of representations of G.

Definition

A smooth representation π of G is supercuspidal if $r_{L,P}^G(\pi) = 0$, for any proper parabolic subgroup P of G.

Remark

An irreducible smooth representation π of G is supercuspidal if and only if it is not a subquotient of a proper parabolically induced representation.

Definition

A complex valued function on G of the form

$$c_{\tilde{v},v} \colon g \mapsto \langle \tilde{v}, \pi(g)v \rangle,$$

for a fixed $(\tilde{v}, v) \in \tilde{V} \times V$, is called a matrix coefficient of π.

Proposition

An irreducible G-representation is supercuspidal if and only if all its matrix coefficients have compact support modulo the centre of G.

Example

Any supercuspidal irreducible representations of $\mathrm{SL}_2(F)$ can be induced irreducibly from an compact open subgroup of $\mathrm{SL}_2(F)$.
Still true for any classical group G if $p \neq 2$, and for arbitrary G if p does divides the order of the Weyl group of G.

Quadratic extensions of F

When p is odd, the field F has three quadratic extensions, and they have the form $F(\sqrt{a})$, where a is ε_F (an element of order $q-1$ of $F^\times$), ϖ_F, and $\varepsilon_F \varpi_F$. The extension $F(\sqrt{a})/F$ is unramified if $a = \varepsilon_F$ and is ramified in the other two cases.

Notation

For any nonsquare element a of $F^\times$, we define

$$\mathrm{sgn}_a(x) := \begin{cases} 1 & \text{if } x \text{ belongs to the image of } \mathrm{N}_{F(\sqrt{a})/F} \\ -1 & \text{otherwise.} \end{cases}$$

For each extension $F(\sqrt{a})/F$ there exists a corresponding family of supercuspidal representations of $\mathrm{SL}_2(F)$. Each family can be parameterized by a nontrivial additive character of F and nontrivial characters χ of the kernel of the norm map $\mathrm{N}_{F(\sqrt{a})/F}$.

Theorem [Harish-Chandra]

Any smooth irreducible π of G occurs as an irreducible component of a parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$, where P is a parabolic subgroup of G with Levi factor L and $\sigma \in \mathrm{Irr}(L)$ is supercuspidal. The G-conjugacy class $(L, \sigma)_G$ of (L, σ) is uniquely determined and is called the supercuspidal support of π.

Definition

An irreducible representation π of G is square integrable modulo the center if $\pi_{|\mathrm{Z}_G}$ is a (unitary) character of Z_G and if the (complex) absolute value of every matrix coefficient of π belongs to $L^2(G/\mathrm{Z}_G)$, that is,

$$\int_{G/\mathrm{Z}_G} |\langle \tilde{v}, \pi(g)v \rangle|^2 \mathrm{d}\bar{g} < \infty \quad \text{for all } v \in V,\ \tilde{v} \in \tilde{V},$$

where $\mathrm{d}\bar{g}$ is a Haar measure on the group G/Z_G.

Notation/Definition

Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$ and $\mathbf{B} \supset \mathbf{T}$ a Borel subgroup in $\mathbf{G}$. Let $\Phi(\mathbf{G}, \mathbf{T})$ be the associate root system and Δ a basis of $\Phi(\mathbf{G}, \mathbf{T})$. For $I \subset \Delta$, we denote by $\mathbf{L}_I \supset \mathbf{T}$ and $\mathbf{P}_I \supset \mathbf{B}$ the Levi and parabolic subgroups associated to I, and write simply $i_{L_I}^G$ for the normalized version of i_{L_I,P_I}^G, and similarly for parabolic restriction.

Definition

We define the following operator in the Grothendieck group of the category of smooth representations of G :

$$D_G := \sum_{I \subset \Delta} (-1)^{|I|} i_{L_I}^G \circ r_{L_I}^G.$$

Theorem

D_G is an involution and preserves irreducibility, up to sign : if π is irreducible then $\pm D_G(\pi)$ is irreducible.

The Steinberg representation

Let triv_G denote the trivial representation of G. The representation

$$\mathrm{St}_G := D_G(\mathrm{triv}_G) = \sum_{I \subset \Delta} (-1)^{|I|} i_{L_I}^G(\mathrm{triv}_{L_I})$$

is an irreducible representation in the discrete series of G.

Definition

Let (π, V) be a smooth representation of G. Let $C_c^\infty(G)$ denote the space of functions $G \to \mathbb{C}$ which are compactly supported and locally constant. Given $f \in C_c^\infty(G)$, the operator $\pi(f)\colon V \to V$ is defined by

$$\pi(f)v := \int_G f(g)\pi(g)v\mathrm{d}g, \quad \text{for } v \in V.$$

Here $\mathrm{d}g$ denotes a left Haar measure on G.

Proposition

A smooth representation (π, V) of G is admissible if and only if $\pi(f)$ has finite rank for every $f \in C_c^\infty(G)$.

If π is an admissible representation, we can define a linear functional on $C_c^\infty(G)$ that is analogous in some ways to the character of a finite-dimensional representation of a finite group.

Definition

Let (π, V) be an admissible representation of G. The character of π is the distribution Θ_π on G defined by $\Theta_\pi(f) := \mathrm{tr}(\pi(f))$, for $f \in C_c^\infty(G)$.

Theorem [Harish-Chandra]

There exists a function $\theta_\pi \colon G \to \mathbb{C}$ such that

$$\Theta_\pi(f) = \int_G f(g)\theta_\pi(g)\mathrm{d}g \quad \text{for all } f \in C_c^\infty(G).$$

The function θ_π is a class function : $\theta_\pi(gxg^{-1}) = \theta_\pi(x)$, for all $x, g \in G$.

Definition

Let $\mathcal{D}^G \colon G \to \mathbb{C}$ denote the Weyl discriminant function defined by letting $\mathcal{D}^G(\gamma)$ be the coefficient of T^d, where $d = \mathrm{rank}(G)$, in the polynomial $\det(T + 1 - \mathrm{Ad}(\gamma))$. An element $\gamma \in G$ is called regular if $\mathcal{D}^G(\gamma) \neq 0$. Let G_{reg} denote the set of regular elements in G.

Example

For $G = \mathrm{SL}_2(F)$, we have

$$\mathcal{D}^G \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d)^2 - 4.$$

Remark

The function θ_π is usually viewed as a function on the regular set G_{reg}. Since G_{reg} is an open dense subset of G, we may consider θ_π as a function on G by setting $\theta_\pi(g) = 0$ for all elements g in the complement of G_{reg} - this complement is a set of measure zero.)

Let $\mathfrak{g}$ denote the Lie algebra of G and let B be a fixed symmetric, nondegenerate, bilinear, G-invariant form on $\mathfrak{g}$.

Fix a depth zero additive character ψ on F, that is, $\psi_{\mathfrak{p}_F} \equiv 1$ and $\psi_{\mathfrak{o}_F} \not\equiv 1$. For $f \in C_c^\infty(\mathfrak{g})$, define the Fourier transform of f by

$$\hat{f}(X) := \int_{\mathfrak{g}} f(Y) \cdot \psi(B(X, Y)) \, \mathrm{d}Y,$$

for $X \in \mathfrak{g}$ and $\mathrm{d}Y$ the self-dual measure on $\mathfrak{g}$.

Fact

Let $\mathcal{O}$ be an orbit in $\mathfrak{g}$ under the adjoint action of G on $\mathfrak{g}$. Then $\mathcal{O}$ carries a G-invariant Radon measure $\mu_{\mathcal{O}}$ (in $\mathfrak{g}$).

Definition

We write $\hat{\mu}_{\mathcal{O}}(f) := \mu_{\mathcal{O}}(\hat{f})$ for any $f \in C_c^\infty(\mathfrak{g})$.

Theorem [Harish-Chandra]

The distribution $\hat{\mu}_{\mathcal{O}}$ is represented by a locally integrable function on $\mathfrak{g}$ which we again denote by $\hat{\mu}_{\mathcal{O}}$. Furthermore, $\hat{\mu}_{\mathcal{O}}$ is locally constant on $\mathfrak{g}_{\mathrm{reg}}$ and $|\mathcal{D}^{\mathfrak{g}}(X)|^{1/2} \cdot \hat{\mu}_{\mathcal{O}}$ is locally bounded on $\mathfrak{g}$, where $\mathcal{D}^{\mathfrak{g}}$ denotes the Weyl discriminant function on $\mathfrak{g}$.

Example

For $G = \mathrm{SL}_2(F)$, we take $B(X, Y) = \mathrm{tr}(XY)$. We have

$$\mathcal{D}^{\mathfrak{g}} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 4(a^2 + bc).$$

Every nilpotent orbit of $\mathfrak{sl}_2(F)$ is of the form $\mathrm{Ad}(g)X_a$ where $X_a = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, with $a \in \{0, 1, \varepsilon_F, \varpi_F, \varpi_F\varepsilon_F\}$.