Propagation of randomness, Gibbs measures and random tensor's theory for NLS

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PDE and Randomness

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The periodic NLS

$$(i\partial_t + \Delta) u = \pm |u|^{p-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d. \qquad \text{(NLS)}$$

with Hamiltonian

$$H[u](t) := \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 \, dx \pm \frac{1}{p+1} \int_{\mathbb{T}^d} |u|^{p+1} \, \mathrm{d}x = H[u](0).$$

and mass $m(u) := \int_{\mathbb{T}^d} |u|^2 dx$ conservation.

- p is odd; the sign only matters for the global dynamics or Gibbs measure.
- Recall the scaling critical threshold: $\boxed{s_{cr} := \frac{d}{2} - \frac{2}{p-1}}$
- Theorem (Local-in-time well-posdedness for NLS on $\mathbb{T}^d$)

  Assume $s_{cr} \geq 0$, then NLS is LWP for data in H^s provided that $s > s_{cr}$.

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  (Bourgain '93, Bourgain-Demeter '15 $\to$ Strichartz estimates on tori.)
- How about $s = s_{cr} = 0 \leftrightarrow L^2$; say cubic NLS on $\mathbb{T}^2$? Unknown!
- If $s < s_{cr}$ ill-posedness may occur (Christ-Colliander-Tao, others ...).

Random Data Theory of NLS

What happens generically? Instead of individual solutions, are interested in the family of solutions that are distributed according to some canonical law (e.g. Gaussian law). Consider NLS with the following canonical random data:

$$u^\omega(0) = f(\omega) = \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{\langle k\rangle^\alpha} e^{ik\cdot x}, \qquad \boxed{\alpha := s + \frac{d}{2}} \tag{ID}$$

where $\{g_k\}$ are i.i.d. complex Gaussian[1] r.v., $\mathbb{E}g_k = 0$, $\mathbb{E}|g_k|^2 = 1$.
Formally the Gaussian measure:

$$d\rho_\alpha \sim \exp\left(-\|u\|^2_{H^\alpha}\right) \cdot \prod_{x\in\mathbb{T}^d} du(x).$$

$d\rho_\alpha$ is supported in $H^{s-}(\mathbb{T}^d) := \bigcap_{\varepsilon>0} H^{s-\varepsilon}(\mathbb{T}^d)$. Almost surely in ω the random initial data $f(\omega)$ belongs to H^{s-} $\boxed{s := \alpha - \frac{d}{2}}$.

[1] For $\rho_k = |g_k|$ can also consider $\eta_k := g_k\rho_k^{-1}$, uniformly distributed in the unit circle $|z| = 1$ (random phase).

Almost sure local well-posedness

By switching to almost-sure point of view, one can cross the scaling barrier and get almost sure local well-posedness for values $s < s_{cr}$.

- Historically people only know weak solutions in the supercritical case (no uniqueness, and we know little about this).
- In groundbreaking work, Bourgain '96 showed that if we consider canonical random data in some deterministically supercritical space $s < s_{cr}$ then almost surely one can get strong solutions!.
- A key point: for random initial data $\longrightarrow$ better linear and nonlinear estimates than those for arbitrary functions of the same regularity $\longleftarrow$ (linear and multilinear) large deviation estimates and other type of random matrix estimates.

Fundamental questions

Despite numerous follow-up works to Bourgain's work, fundamental questions remained open:

- What's the optimal value of s for almost-sure LWP to hold?
- How does a given initial random data get transported by the NLS flow?
  - If it is Gaussian initially, how does this Gaussianity propagate?
  - What's the description of the solution beyond the linear evolution?

[2] see Frölich-Knowles-Schlein-Sohinger; Lewin-Nam-Rougerie; Sohinger in context of limits of grand canonical thermal states in many-body quantum mechanics

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Such questions are important in many topics such as:

— invariance of Gibbs measures (statistical mechanics; CQFT; SPDE); lie nicely at the intersection of PDE and statistical physics[2].

— (weak) wave turbulence (density and statistics of the interacting waves).

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Invariant Gibbs measures

- If $\boxed{\alpha = 1}$ we have random data:

$$u(0) = f(\omega) = \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle} e^{ik\cdot x}, \qquad \boxed{s := 1 - \frac{d}{2}}$$

  the typical element in the support of the Gibbs measure associated to NLS. In 2D and 3D such Gibbs measures are supported on distributions $H^{0-}(\mathbb{T}^2)$ and $H^{-\frac{1}{2}-}(\mathbb{T}^3)$ respectively.

- From the Hamiltonian structure of NLS with $\mathcal{H}(u)$ the Gibbs measure can be formally defined as

$$d\mu \sim e^{-\mathcal{H}(u)} \cdot \prod_{x \in \mathbb{T}^d} du(x) \sim \underbrace{\exp\left(-\frac{1}{p+1} \|u\|_{L^{p+1}}^{p+1}\right)}_{\text{weight}} \, d\rho_1$$

- In some cases this measure can be rigorously defined as a weighted Gaussian measure. Construction -and its properties under various dynamics- is a major problem in statistical mechanics and CQFT (intimately related to the so-called Φ^4 model when $p = 3$).

  Glimm-Jaffe, Lebowitz-Rose-Speer, Simon, Nelson, Aizenman, Fröhlich, ...

Construction

- $d = 1, 2$: construction can be done for any p.
  - Measure is absolutely continuous w.r.t. Gaussian measure
- $d = 3$: construction can be done for $p = 3$.
  - But measure is not absolutely continuous w.r.t. Gaussian measure! (recent work by Barashkov-Gubinelli $\rightarrow$ better understanding)
- $d \geq 4$: it cannot be done for any p

  (Frölich, Aizenman, Aizenman- Duminil-Copin, ...).

How about Invariance under NLS flow together with existence of global strong solutions on its statistical ensemble.

Formally $d\mu$ can be seen to be invariant under the flow of NLS due to a formal Liouville's theorem and conservation of the (renormalized) Hamiltonian $\mathcal{H}$ . . . .

Difficulty for dynamics: Gibbs msr. supported in $H^{1-d/2-}$ (rough if $d \geq 2$).

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- How about $d = 2$, $p \geq 5$? Open 1996 $\rightarrow$ 2019 (Y. Deng–A.N.–H. Yue) .

  We prove the invariance of Gibbs measure and a.s. existence of strong solutions to NLS on $\mathbb{T}^2$ and any odd power nonlinearity $p \geq 5$.

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- How about $d = 3$, $p = 3$? Open, and very hard! Why?

  $s_{cr} = \frac{1}{2}$ in this case, so why is it so much harder than $d = 2$, $p = 5$ which also has $s_{cr} = \frac{1}{2}$?

Answers

We find the optimal value

$$\boxed{s_{pr} := -\frac{1}{p-1}} \leq s_{cr},$$

the critical index in probabilistic scaling, as the threshold for NLS with random data. Heuristically s_{pr} is based on the basic idea of the square root cancellation of sums of independent random variables.

[3] Assume $(d, p) \neq (1, 3)$.

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In particular:

- We obtain local-in-time strong solutions for random data in the full probabilistic subcritical range $s > s_{pr}$ for any dimension[3] and give a precise description of the solution in terms of multilinear gaussians.

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  - Key: Support of the Gibbs measure $\mathrm{d}\mu$ in 2D is H^{0-}, which is *probabilistically subcritical* for any such p.

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    - Key: Support of the Gibbs measure $\mathrm{d}\mu$ in 2D is H^{0-}, which is *probabilistically subcritical* for any such p.
- We also improve Bourgain's result for the 3D Hartree-NLS Gibbs measure (more on this at the end).

[3] Assume $(d, p) \neq (1, 3)$

Probabilistic Scaling

Start with a frequency scale N and random initial data

$$u_0 = f = N^{-\alpha} \sum_{|k| \sim N} g_k(\omega) e^{ik\cdot x} \qquad \alpha = s + \frac{d}{2}$$

then f has unit size in H^s. If NLS is a.s. LWP then the second iteration

$$u^{(1)}(t) = \int_0^t e^{i(t-s)\Delta}(|e^{is\Delta} f|^{p-1} \cdot e^{is\Delta} f)\, \mathrm{d}s$$

should be bounded in H^s for fixed time t.

Fix $|t| \sim 1$, and for $|k| \sim N$ calculate the Fourier coefficients of $u^{(1)}(t)$,

$$\widehat{u^{(1)}}(t, k) \sim N^{-p\alpha} \sum_{k_1 - \cdots + k_p = k} \frac{1}{\langle \Omega \rangle} \cdot g_{k_1}(\omega) \overline{g_{k_2}(\omega)} \cdots g_{k_p}(\omega),$$

where $|k_j| \sim N$, $\Omega = |k|^2 - |k_1|^2 + \cdots - |k_p|^2$ is the resonance factor, and $\pm$ represent possible complex conjugates.

For simplicity we may restrict to $\Omega = 0$, reducing to the sum

$$\widehat{u^{(1)}}(t,k) \sim N^{-p\alpha} \sum_{\substack{k_1 - \cdots + k_p = k \\ |k_j| \sim N, \Omega = 0}} g_{k_1}(\omega)\overline{g_{k_2}(\omega)} \cdots g_{k_p}(\omega).$$

If we assume $k_1 \neq k_2$, $k_2 \neq k_3$ and so on, these r.v. are independent. This leads to the square root cancellation (gain). We conclude that

$$|\widehat{u^{(1)}}(t,k)| \sim N^{-p\alpha} \Bigg(\sum_{\substack{k_1 - \cdots + k_p = k \\ |k_j| \sim N, \Omega = 0}} 1 \Bigg)^{1/2} \sim N^{-p\alpha} N^{(pd-d-2)/2}.$$

by dimension counting.

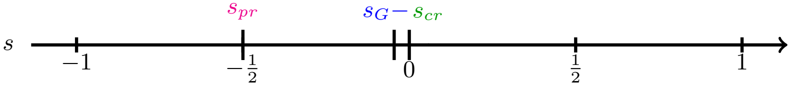
Then, $u^{(1)}(t)$ is bounded in H^s if and only if

$$-p\alpha + \frac{pd-d-2}{2} + \overbrace{s + \frac{d}{2}}^{=\alpha} \leq 0 \Leftrightarrow s \geq -\frac{1}{p-1} := s_{pr}.$$

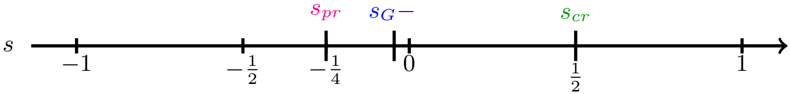
Probabilistic v. deterministic scaling

Recall $s_{pr} = -\frac{1}{p-1}$ and let $s_G = 1 - \frac{d}{2}$ Gibbs measure supported in $H^{s_G-}(\mathbb{T}^d)$.

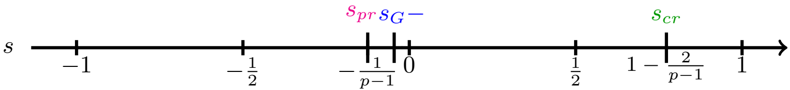
- $d = 2, p = 3$



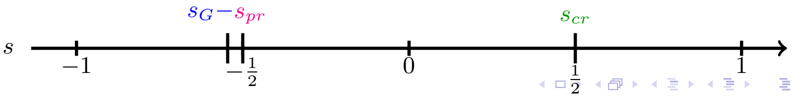
- $d = 2, p = 5$



- $d = 2, p$ large



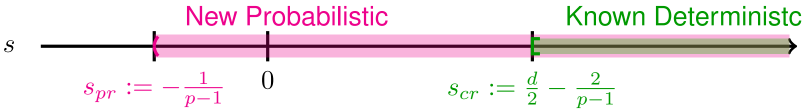
- $d = 3, p = 3$



Main result I

Theorem 1 (Deng-N.–Yue 2020)

Let $p \geq 3$ odd and for $s > -\frac{1}{p-1} = s_{pr}$, and let $\boxed{\alpha = s + \frac{d}{2}}$. Consider (NLS) on $\mathbb{T}^d$ under suitable renormalization, with α-random initial data (ID).
Then almost surely in ω, there exists a strong local solution that is unique in a suitable sense.
Furthermore, this solution has an explicit expansion in terms of multilinear Gaussians with adapted random tensor coefficients.



Bourgain: $\mathbb{T}^2$, $p = 3$ ($s_{cr} = 0$), Prob. LWP for some $s < 0$.

Remarks

- The renormalization we need is the Wick ordering. An infinite L^2 mass implies that the potential energy is almost-surely infinte and the nonlinearity $|u|^{p-1}u$ of (NLS) does not make sense a.s. as distributions.

  This 'infinity' has to be removed by suitably renormalizing the nonlinearity.

- Uniqueness is in the sense that, our solution is the unique limit for all possible choices of canonical approximations (or regularizations).

- Note one barely misses a.s. LWP for the $d = 3$ cubic NLS in $H^{-\frac{1}{2}-}$ when $s_{pr} = -\frac{1}{2}$.

Main Result II. Long time.

Our second result concerns *well prepared* smooth data –such as that arising in derivation of WKE in wave turbulence theory – and long time existence of solutions and predicts the time of the first energy cascade, which is consistent with wave turbulence theory and expected to be sharp.

Theorem 2 (Deng-N.–Yue 2020)

Fix (s, α) as before, let N be dyadic and ϕ be Schwartz. Let u solve (NLS) with *random homogeneous data on* $\mathbb{T}^d$ defined by

$$u(0) = f(\omega) = N^{-\alpha} \sum_k \phi\left(\frac{k}{N}\right) g_k(\omega) e^{ik\cdot x}.$$

Then, with high probability, there is no energy cascade between Fourier modes (i.e. $|\widehat{u}(t,k)|^2 \approx |\widehat{u}(0,k)|^2$ with negligible error for large N) up to the time

$$T = N^{(p-1)(s-s_{pr})-}.$$

With high probability, $\|u(0)\|_{H^s} \sim 1$.

Main result III. Invariance and global strong solutions.

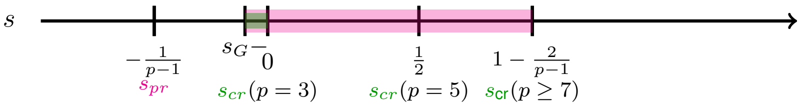
Our third main result[4] (proved before Theorem 1) is

Theorem 3 (Deng-N.–Yue 2019)

Let $d = 2$ and $p \geq 3$ odd. Then the renormalized NLS is almost surely globally well-posed on the support of the Gibbs measure $\mathrm{d}\mu$ (which is in H^{0-}). The global nonlinear flow Φ_t maps a full measure set Σ to itself, forms a one-parameter group (i.e. $\Phi_{t+s} = \Phi_t \Phi_s$), and keeps the Gibbs measure $\mathrm{d}\mu$ invariant under the flow:

$$\mu(E) = \mu(\Phi_t(E))$$

for any Borel set $E \subset \Sigma$.



[4] Weak solutions (with no uniqueness) that preserve $\mathrm{d}\mu$ (in some sense) were previously obtained by Oh-Thomann.

Bourgain's method ('96)

Before describing our new methods let us review the existing approach thus far for studying random data local well-posedness due to Bourgain.

Bourgain considered the cubic (Wick ordered) NLS equation on $\mathbb{T}^2$

$$iu_t + \Delta u \;=\; \underbrace{\text{“}\,|u|^2 u - 2u\Big(\int |u|^2 dx\Big)\,\text{”}}_{\mathcal{C}(u)}$$

with random initial data:

$$u_0^\omega(x) := \sum_{k\in\mathbb{Z}^2} \frac{g_k(\omega)}{\langle k\rangle}\, e^{ik\cdot x}, \qquad x \in \mathbb{T}^2.$$

The initial data (in support of the Gibbs measure) is in H^{0-}, and is thus is deterministically supercritical but probabilistic subcritical:

$$s_{pr} \;=\; -\frac{1}{2} < 0- \;<\; 0 \;=\; s_{cr}.$$

Then

- Bourgain's main idea is to make a linear-nonlinear decomposition, where the linear part is **rough** and **random**, and the nonlinear part is smoother.
- He constructed solutions of form $\boxed{u = e^{it\Delta} f(\omega) + v}$ and showed v has **positive** regularity.
- Idea: Solve the difference initial value problem via a Banach fixed point argument on a ball in **a smoother space**:

$$\begin{cases} iv_t + \Delta v = \mathcal{C}\Big(\underbrace{e^{it\Delta} f(\omega)}_{\text{R:=rough-random}} + \underbrace{v}_{\text{smoother-deterministic=:D}} \Big) \\ v(x,0) = 0, \qquad x \in \mathbb{T}^2 \end{cases}$$

Tools:

- multilinear large deviation estimates
- integer lattice counting estimates $\leftrightarrow$ analytic number theory
- TT^* arguments $\leftrightarrow$ random matrix estimates (correct way to exploit randomness in absence of gain of regularity).

2D NLS Gibbs: what's happening?

- Let us consider $p = 5$, and recall $s_{cr} = \frac{1}{2}$ and following Bourgain write $u = u_{\text{lin}} + v$ then $u_{\text{lin}} \in H^{0-}$ but v can only be put in $H^{\frac{1}{2}-}$ which still (det.) supercritical: one cannot close the estimate by itself.

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- This poor regularity comes <u>only</u> from high-low frequency interactions so we may try to identify a term X from v that is paracontrolled by u_{lin},

$$X = \mathcal{I}\pi_{>}(u_{\text{lin}}, :|u|^{p-1}:) + \text{smoother term}, \qquad \mathcal{I} = (i\partial_t - \Delta)^{-1}$$

  and hope that X behaves like u_{lin} and that $Y := v - X$ is smoother.

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  and hope that X behaves like u_{lin} and that $Y := v - X$ is smoother.
- But to justify this we need some control on the lower frequency parts of $:|u|^{p-1}:$ which itself contain a part of X, that is $:|X|^{p-1}:$ whose regularity is $H^{\frac{1}{2}-}$, supercritical, and there is no way of controlling $:|X|^{p-1}:$ assuming only this.
  - For wave or heat X has higher regularity due to smoothing so the low frequency part above is a nice function and can be place directly into a good function space.

How do we tackle this?

- We need to **zoom in/unveil and invoke the structure** of X. But how can this be done?
  - To justify the behavior of X we need to justify the behavior of $X \longrightarrow$ induction on frequency and justify the behavior of the lower frequency parts of X before justifying the high frequency part of X.

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  - For simplicity let us write (where P_N are Littlewood-Paley projections)

$$P_N X = \sum_{L \ll N} \mathcal{I}(P_N(u_{\text{lin}}) \cdot : |P_L u|^{p-1} :)$$

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  - It is possible to make a recursive reduction. Namely we write $P_N X$ as a (nonlinear) expression of $P_L X$ for $L \ll N$ (up to some remainder terms that presumably can be controlled), then iterate and write $P_L X$ as an expression of $P_R X$ for $R \ll L$, and so on, until the frequency reaches 1.

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  - But this iteration process leads to complicated tree expansion, involving all frequency scales from N to 1: hard to treat simultaneously and the amount of combinatorics is overwhelming. Maybe could try do for $p = 5$ but we seek a different and unified method.

Paradigm shift

- Goal: capture the implicit randomness structure of $P_N X$ in some norm or quantity that **propagates**; i.e. that allows for an induction from frequencies $L \ll N$ to frequency N.

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- Clearly the $H^{\frac{1}{2}-}$ norm of $P_N X$ will not do the job, since this is a supercritical norm. To find the right quantity, we need to **shift point of view** and instead of considering the term

$$P_N X = \sum_{L \ll N} \underbrace{\mathcal{I}(P_N(u_{\text{lin}}) \cdot \ : |P_L u|^{p-1} :)}$$

we consider the **operator**

$$\mathcal{P}_{NL} : y \longrightarrow \mathcal{I}(P_N(y) \cdot \ : |P_L u|^{p-1} :).$$

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$$P_N X = \sum_{L \ll N} \underbrace{\mathcal{I}(P_N(u_{\text{lin}}) \cdot \ : |P_L u|^{p-1} \ :)}$$

we consider the **operator**

$$\mathcal{P}_{NL} : y \longrightarrow \mathcal{I}(P_N(y) \cdot \ : |P_L u|^{p-1} \ :).$$

  - $\mathcal{P}_{NL}$ depends on $P_L u$, which has an implicit random structure $\sim$ para-product structure $\sim$ a projection onto frequency N, followed by a weighted average over smaller scales $L \ll N$, say when applied to the Gaussian free field u_{lin}.

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  - For this reason we will call it a **random averaging operator** (matrix operator), and is the key to propagating the right probabilistic bounds.

Norms and a priori bounds

- The operator $\mathcal{P}_{NL}$, whose coefficients will be independent with the modes of $P_N(u_{\rm lin})$, contains all the randomness information of the low frequency components of u, which is then carried by two operator norms a priori estimates:

$$\|\mathcal{P}_{NL}\|_{\rm OP} \lesssim L^{-\delta_0}, \quad \|\mathcal{P}_{NL}\|_{\rm HS} \lesssim N^{1/2+\delta_1} L^{-1/2}, \qquad (\dagger)$$

  for some $\delta_1 \ll \delta_0 \ll 1$.

- The HS-norm bound guarantees that $P_N X = \sum_{L \ll N} \mathcal{P}_{NL} u_{\rm lin}$ belongs to $H^{1/2-}$ as expected; but as we mentioned we cannot use this directly. The operator norm[5]bound, on the other hand, is the key norm that allow us to propagate.

[5] $\mathcal{P}_{NL}$ viewed as a linear operator between two Hilbert spaces: L^2 or the Fourier restrictions $X^{s,b}$ spaces. Note they're global-in-space objects, consistent with non-local setting of Schrödinger eqs.

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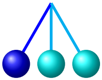
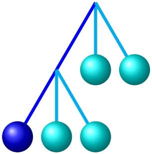
- If we had general functions in $X^{\frac{1}{2}-,\frac{1}{2}+}$ as low inputs then we would never be able to prove these two estimates. So these really capture the randomness structure of the low part. They are also tractable and simple to use.

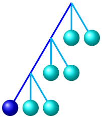
[5] $\mathcal{P}_{NL}$ viewed as a linear operator between two Hilbert spaces: L^2 or the Fourier restrictions $X^{s,b}$ spaces. Note they're global-in-space objects, consistent with non-local setting of Schrödinger eqs.

RAO Revisited $\longrightarrow$ (1,1) random tensors ($p = 3$)

Let $\mathcal{I}$ = Duhamel operator. Denote  $:= e^{it\Delta} f_N(\omega)$ and  $:= u_{N^\delta}$. Naturally we can also define

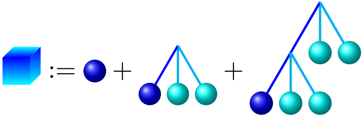
 $:= \mathcal{IC}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta})$,

 $:= \mathcal{IC}\big(\mathcal{IC}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}), u_{N^\delta}, u_{N^\delta}\big)$,

and so on  $, \cdots .$

The sum of these trees forms on an infinite series of trees:

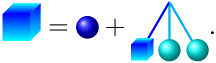
$$\Psi_{N,N^\delta} = $$



$$+ \cdots .$$

which is equivalent to the para-linearized equation:

$$\Psi_{N,N^\delta} = u_{\text{lin}} + \mathcal{P}_{N,N^\delta}(\Psi_{N,N^\delta}) \Longleftrightarrow$$



By solving this equation, we have

$$\Psi_{N,N^\delta} = (\mathrm{Id} - \mathcal{P}_{N,N^\delta})^{-1}(u_{\text{lin}}) := \mathcal{P}_N(u_{\text{lin}}),$$

and that the k-th Fourier mode of Ψ_{N,N^δ} is in the following form:

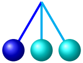
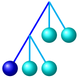
$$\boxed{\mathcal{F}\big(\Psi_{N,N^\delta}\big)(k) = \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha}}$$

where h_{kk_1} is the $(1,1)$ random tensor (matrix); indep. of $g_{k_1}(\omega)$.

The full ansatz $\rightarrow$ 2D Gibbs measure problem

Ansatz for the solution is

$$u = \sum_N \Psi_{N,N^\delta} + \text{remainder}$$

$$= u_{\text{lin}} + \Big(\sum_N$$  $$+$$  $$+ \cdots\Big) + \text{remainder}$$

$$= u_{\text{lin}} + \sum_N \mathcal{P}_N(u_{\text{lin}}) + \text{remainder}$$

where the remainder belongs to the subcritical space H^{1-}.

Expanding the solution u in Fourier space, where $u_k(t) := \widehat{u}(t,k)$, we have

$$\boxed{u_k(t) = \frac{g_k(\omega)}{\langle k\rangle^\alpha} + \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1\rangle^\alpha} + (\text{remainder})_k} \qquad \text{(FC-RAO)}$$

where h_{kk_1} is the $(1,1)$ random tensor (matrix) independent from g_{k_1} and containing all the randomness information of the low frequency components of the solution u and prove suitable operator norm estimates for h_{kk_1}.

The theory of random tensors

- We now focus on the proof of Theorem I which relies on the theory of random tensors.
- Theorem II on long time solutions is a special case.
- The method of random averaging operators (RAO) lays out the foundation for the more general random tensors' theory (RTT). RAO has a much simpler form than random tensors, and suffices in many cases where one is not too close to probabilistic criticality.
- For Theorem I we need higher order expansions[6] which naturally lead to the multilinear expressions as well as the associated random $(q, 1)$, tensors $h = h_{kk_1 \cdots k_q}$, which depend on the low frequency components of the solution.
- For simplicity we henceforth assume $p = 3$.

[6] To find the ansatz we iterate/keep expanding the nonlinearity using the equation itself but we stop expanding the factor as soon as we hit a "low frequency" input -say- with $L < N^{\delta}$, $0 < \delta \ll 1$. Assume we expand up so some high order D (arbitrary and fixed).

Higher order random tensors: how they arise.

Ansatz: expand the solution u to NLS in Fourier space; and write $u_k(t) := \widehat{u}(t,k)$ as

$$u_k(t) = \sum_{q \leq D} \sum_{k_1, \cdots, k_q} h_{kk_1 \cdots k_q}(t) \prod_{j=1}^{q} \frac{g_{k_j}^{\pm}(\omega)}{\langle k_j \rangle^{\alpha}} + (\text{remainder})_k \qquad \text{(EXP)}$$

where $(g^+, g^-) := (g, \overline{g})$ and assume there is no *pairing*, i.e. $k_{j'} \neq k_j$ if the corresp. $\pm$ signs are the opposite in the given q-tuple. If $|k_j| \sim N_j$ and $\max(N_1, \ldots, N_q) = N$, then $N_j \geq N^{\delta}$ all $j = 1, \ldots q \longrightarrow$ independence.

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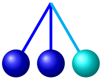
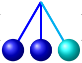
- These quantities $h_{kk_1\dots k_q}$, where t is viewed as a parameter, are the random tensors which will be the main subject of study.
- The convergence of the expansion (EXP) is completely determined by the properties and estimates of these tensors.
- The high-order tensors $h_{kk_1\cdots k_q}$ are from the high-order iteration trees, as in the following examples:

Simple example (p=3): (2,1) tensor term



$$:= \mathcal{IC}(e^{it\Delta} f_{N_1}(\omega),\, e^{it\Delta} f_{N_2}(\omega),\, u_{N^\delta})$$

where $\mathcal{I}$ is the Duhamel operator, $N = \max(N_1, N_2)$ and $N_1, N_2 > N^\delta$. Then (modulo details about the temporal frequency), the k-th Fourier mode is



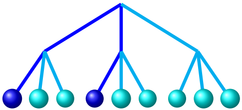
$$\mathcal{F}(\,\cdot\,)(k) \sim \sum_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}} \frac{g_{k_1}(\omega)}{\langle k_1\rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2\rangle^\alpha} \widehat{u}(k_3)$$

$$= \sum_{k_1,k_2} \underbrace{\left(\sum_{|k_3|\le N^\delta} \mathbf{1}_{\left\{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}\right\}} \widehat{u}(k_3) \right)}_{h_{kk_1k_2}} \frac{g_{k_1}(\omega)}{\langle k_1\rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2\rangle^\alpha}$$

where $|k_1| \sim N_1$, $|k_2| \sim N_2$ and $|k_3| \le N^\delta$. Note that here $h_{kk_1k_2}$ is a (2,1) random tensor -say- maps $k_1, k_2 \to k$.

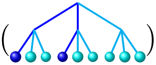
One more example for (2,1) tensor term

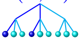
$$\mathcal{IC}\Big(\mathcal{IC}(e^{it\Delta}f_{N_a},u_{N^\delta},u_{N^\delta}),\mathcal{IC}(e^{it\Delta}f_{N_b},u_{N^\delta},u_{N^\delta}),\ \mathcal{IC}(u_{N^\delta},u_{N^\delta},u_{N^\delta})\Big)$$

$:=$ 

where $N = \max(N_a, N_b)$ and $N_a, N_b > N^\delta$.

- Similarly

$\mathcal{F}($  $)(k) = \sum_{\substack{|a|\sim N_a \\ |b|\sim N_b}} h_{kab} \cdot \frac{g_a(\omega)}{\langle a\rangle^\alpha}\,\frac{\overline{g_b(\omega)}}{\langle b\rangle^\alpha}$

- Note that here h_{kab} is a $(2,1)$ random tensor (which maps $a,\ b\ \to\ k$) associated to the term .

Random tensors framework

- RTT allow us to get a handle on the exploding complexity that arises from the higher order tree iterations.

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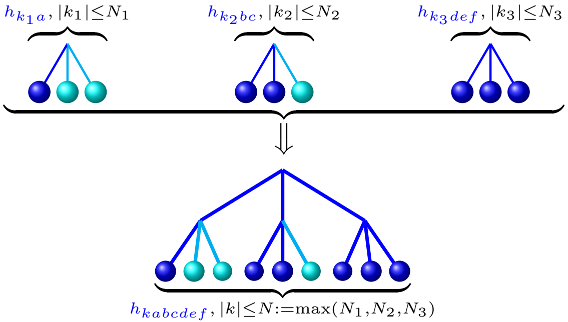
- We develop an algebraic theory which focuses on the structure of the tensors h occurring in (EXP) and how they are built from smaller tensors; using certain operations such as tensor products, contractions, etc. gives rise to two algebraic operations: merging and trimming.

- We also develop the analytic theory, which entails choosing suitable norms for the tensors $h = h_{kk_1 \ldots k_q}$ which behave well with our algebraic theory. We prove several multilinear estimates to provide suitable bounds for the (merged and trimmed) tensors and remainder in (EXP).

Algebraic Theory: Merging



- No pairing case. $\mathcal{IC}(\text{“}(1,1)\text{”}, \text{“}(2,1)\text{”}, \text{“}(3,1)\text{”})$ yields the $(6,1)$ tensor:

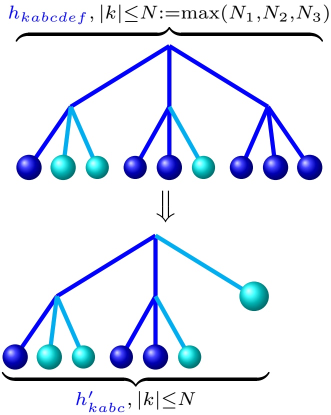
$$h_{kabcdef} = \sum_{k_1, k_2, k_3} \underbrace{(h_{\mathrm{b}})_{kk_1k_2k_3}}_{\text{base tensor}} \cdot h_{k_1 a}\, h_{k_2 bc}\, h_{k_3 def}.$$

- Pairing case: $a = b$. The corresponding merged $(4,1)$ tensor in this case is:

$$h_{kcdef} = \sum_{a=b} h_{kabcdef} \frac{|g_a(\omega)|^2}{\langle a \rangle^{2\alpha}}.$$

Algebraic Theory: Trimming

For example, in the no pairing case if -say- after merging $N_3 < N^\delta$ and $N_1, N_2 \geq N^\delta$, we need to trim the tree to guarantee independence.



where

$$h'_{kabc} := \sum_{d,e,f} h_{kabcdef} \frac{g_d(\omega)}{\langle d \rangle^\alpha} \frac{\overline{g_e(\omega)}}{\langle e \rangle^\alpha} \frac{g_f(\omega)}{\langle f \rangle^\alpha}$$

Analytic theory

- A crucial component of the RTT is the choice of norms for the tensors $h = h_{kk_1\ldots k_q}$ that behave well with the algebraic process of merging and trimming.
- It turns out that the suitable (and only) norms we need are the operator norms when the tensors are viewed as linear mappings from a function of part of the variables to a function of the remaining variables.
- For example for a tensor $h = h_{kxyz}$ we define

$$\|h\|^2_{kx\to yz} := \sup\left\{\sum_{y,z}\left|\sum_{k,x} h_{kxyz}\cdot z_{kx}\right|^2 : \sum_{k,x}|z_{kx}|^2 = 1\right\}$$

- In some instances, we just use the ℓ^2 norm of h in all its variables (Hilbert-Schmidt norm), for example for $h = h_{ab}$, we have

$$\|h\|^2_{ab} = \sum_{a,b}|h_{ab}|^2.$$

Analytic theory

- For example, for merged tensors we have

$$\mathfrak{h}_{bczw} = \sum_{a,e,f} (h^1)_{abc} (h^2)_{aef} (h^3)_{efzw}$$

we have the following multilinear estimate

$$\|\mathfrak{h}\|_{bz \to cw} \leq \|h^1\|_{ab \to c} \|h^2\|_{ef \to a} \|h^3\|_{z \to wef}$$

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- Similarly for trimmed tensors:

$$h'_{kxz} = \sum_{yw} h_{kxyzw} \cdot g_y(\omega)\overline{g_w(\omega)},$$

where the random tensor $h = h_{kxyzw}$ is independent with g_y and g_w. we have with high probability that

$$\|h'\|_{kx\to z} \lesssim N^{\varepsilon} \max\big(\|h\|_{kxyw\to z}, \|h\|_{kxy\to zw}, \|h\|_{kxw\to zy}, \|h\|_{kx\to zyw}\big)$$

where N is the max size of $kxyzw$, and $\varepsilon > 0$ is arbitrarily small.

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where N is the max size of $kxyzw$, and $\varepsilon > 0$ is arbitrarily small.

  - Proof goes back to Bourgain's '96 paper and relies on **high order** TT^* argument and multilinear estimates above. Even in the simplest of cases it seems nontrivial to find a direct proof! (study of random matrices with general Gaussian entries)

To conclude

Armed with both the algebraic and analytic theory of the RTT we can go back to NLS and analyze the norms of the tensors appearing in

$$\boxed{u_k(t) = \sum_q \sum_{k_1,\cdots,k_q} h_{kk_1\cdots k_q}(t) \prod_{j=1}^{q} \frac{g_{k_j}^{\pm}(\omega)}{\langle k_j\rangle^{\alpha}} + (\text{remainder})_k} \qquad \text{(EXP)}$$

These bounds are in fact quite simple. We aim at proving essentially[7] that

$$\|h_{kk_1\dots k_q}\|_{kk_1\dots k_r\to k_{r+1}\dots k_q} \lesssim \prod_{j=1}^{q} N_j^{\beta}\,(\max_{r+1\le j\le q} N_j)^{-\beta} \qquad \text{(TB)}$$

for any r, where $\langle k_j\rangle \sim N_j$ and $\beta \equiv \alpha-$.

[7] Proof is by induction based on framework above, counting lemmata and a delicate selection algorithm to exploit the flexibility in (order in which we estimate) the multilinear merging estimates.

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for any r, where $\langle k_j\rangle \sim N_j$ and $\beta \equiv \alpha-$.

Moreover we prove a Fourier weighted estimate that localizes h as a multilinear Fourier multiplier; i.e. in the support of $h_{kk_1\ldots k_q}$ we have that

$$k \approx \pm k_1 \cdots \pm k_q$$

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Periodic Hartree NLS equation on $\mathbb{T}^3$

$$(i\partial_t + \Delta)u \;=: (|u|^2 * V)u : \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \qquad \text{(H-NLS)}$$

where now the Hamiltonian is

$$\mathcal{H}(u)(t) = \int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2} : |u|^2 (V * |u|^2) : \; \mathrm{d}x.$$

- V real, even and so is $\widehat{V}$, V nonnegative, $V_0 = 1$ and V_k behaves like the Fourier multiplier $\langle k \rangle^{-\beta}$. Typical example is the Bessel potential $V(x) = c_\beta |x|^{-(3-\beta)}$, $V(x) \gtrsim 1$ and smooth away from the origin.
- The (deterministic) scaling critical threshold is

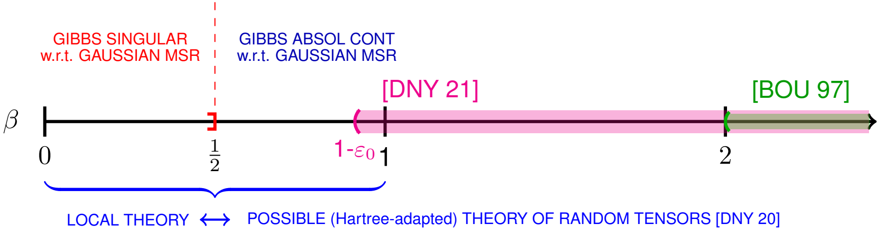
$$s_{cr} := \frac{1-\beta}{2}.$$

For Hartree NLW Bringmann ('20) proved the invariance of Gibbs measure and existence of global strong solutions for $\beta > 0$ and Oh-Okamoto-Tolomeo ('20) proved a similar result for the Hartree SNLW for $\beta > 1/2$.

Main result IV. Hartree NLS equation on $]\mathbb{T}^3$

Theorem 4 (Deng–N.–Yue 2021)

For $\beta > 1 - \varepsilon_0$ ($\varepsilon_0 > 0$ fixed small number) we have the invariance of the Gibbs measure associated to (H-NLS) and the existence of global strong solutions in its statistical ensemble ($1 = \alpha$–random data).



	s_{pr}	s_G	PROBABILISTICALLY	s_{det}
3D NLS	$-\frac{1}{2}$	$-\frac{1}{2}-$	CRITICAL	$\frac{1}{2}$
3D HARTREE NLS	$-\min(\frac{1+\beta}{2}, 1)$	$-\frac{1}{2}-$	SUBCRITICAL ($\beta > 0$)	$\frac{1-\beta}{2}$

Modified RAO adapted to Hartree

Now the full ansatz of the solution to (H-NLS) is:

$$u = \underbrace{u_{\text{lin}}}_{\in H^{-\frac{1}{2}-}} + \underbrace{\sum_N \mathcal{P}_N u_{\text{lin}}}_{\in H^{0-}} + \underbrace{\sum_N \rho_N}_{\in H^{\frac{1}{2}-\varepsilon_1-\varepsilon_2}} + \underbrace{w}_{\in H^{\frac{1}{2}-\varepsilon_1}}, \qquad \text{(RAO2)}$$

- $\mathcal{P}_N u_{\text{lin}}$ is the similar RAO term as in the ansatz of (NLS).
- ρ_N is a modified RAO term which arises from a 'critical' component (explained later) in the Hartree NLS equation. Intuitively,

$$(i\partial_t + \Delta)\rho_N = \Big(P_{\lesssim N^{\varepsilon}}(|u_N|^2 - |u_{\frac{N}{2}}|^2)\Big) * V \cdot \rho_N.$$

- Here $1-\beta \ll \varepsilon_2 \ll \varepsilon_1$, which ensures that both ρ_N and the remainder w are in the deterministic subcritical space ($\in H^{\frac{1-\beta}{2}}$).
- If the frequency of $|u_N|^2 - |u_{\frac{N}{2}}|^2$ is very small (say ~ 1), then the potential V does not lead to any gain of derivatives, and this particular term in fact exhibits a probabilistically critical feature.

The special term ρ_N: a 'critical' component

To see this, for simplicity, let us consider a simplified term:

$$\mathcal{I}\mathcal{N}_{cr}(P_N(u_{\mathrm{lin}}), u_{N/2}, w) =: \mathcal{L}(w),$$

where $\mathcal{N}_{cr}$ is this 'critical' portion of the nonlinearity; i.e.

$$\mathcal{N}_{cr}(v_1, v_2, v_3) := (\Pi_1(v_1\overline{v_2}) * V) \cdot v_3.$$

$\mathcal{N}_{cr}$ behaves like a cubic nonlinearity since essentially the potential V has no effect. In other words, $\mathcal{N}_{cr}$ behaves as a part of the nonlinearity in the 3D cubic NLS Gibbs measure problem, which is probabilistic critical.

Analytically:

- If $w \in H^s$, then $\mathcal{L}(w)$ can be shown to be only in $H^{s-\varepsilon_2}$, hence
  - ρ_N, which satisfies $\rho_N = \mathcal{L}(\rho_N)$, cannot be treated as a part of the smooth remainder w.
  - $\mathcal{L}$ cannot be viewed as $\mathcal{P}_N$, a regular RAO either (structure and $\mathcal{P}_N(w) \in H^s$ is in remainder.)

Final Remarks: Parabolic v.s. dispersive

- There has been spectacular progress in the study of singular SPDE -say- the stochastic heat equation with spacetime white noise thanks to the regularity structures' theory of Hairer and the para-controlled calculus of Gubinelli-Imkeller-Perkowski.
- The stochastic heat equation has been solved in the full subcritical range relative to the parabolic scaling: $s > s_{pa} = -\frac{2}{p-1} = s_{cr} - \frac{d}{2}$ (Bruned-Chandra-Chevyrev-Hairer, Bruned-Hairer-Zambotti, Chandra-Hairer; Chandra-Moinat-Weber).
- Note that $s_{cr} \geq s_{pr} > s_{pa}$. The first inequality shows the effect of randomness, and second exhibits the difference between heat and Schrödinger equations.
- Because of the different scaling and other reasons (lack of smoothing, non-locality of fundamental solutions, etc.), the existing parabolic theories cannot be applied to dispersive equations. Note $S(t) := e^{it\Delta}$ preserves L^2 $\rightarrow$ nice theory in H^s. But it **not** bounded on any other L^p or $W^{s,p}$ norm $\rightarrow$ not a nice theory in e.g. L^∞ or Hölder spaces.
- However, what is important from these theories is that they both feature the study of high-low interactions.
- In the regularity structures' theory the solution is expanded as a Taylor series locally in space involving multilinear Gaussians.
- Our method of treating dispersive equations also involves analyzing high-low interactions and multilinear expansions, except that such expansions are performed in the Fourier space.

Many thanks for your attention!!