Segal's axioms and bootstrap for Liouville theory.

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Quantum Field Theory

(Euclidean) QFT

- Random fields $\Psi(x)$, $x \in M$, M manifold, e.g. $\mathbb{R}^d$
- Expectation $\langle \cdot \rangle$
- Correlation functions $\langle \prod_{i=1}^{N} \Psi(x_i) \rangle$ well defined $x_i \neq x_j$
- Correlations covariant under symmetries of M
- Axiomatizations:
  - Osterwalder-Schrader axioms $M = \mathbb{R}^d$, euclidean symmetry
  - CFT axioms $M = \mathbb{R}^d$, conformal symmetry
  - Belavin-Polyakov-Zamolodchikov and Segal's axioms $M = \mathbb{C}$ or Riemann surface

Conformal Field Theory (CFT)

Euclidean QFT models **statistical physics**

At **critical temperature** such systems have **conformal symmetry** and the QFT is **conformal field theory**

This extra symmetry gives rise to strong constraints on correlation functions via **conformal bootstrap**

In 2 dimensions bootstrap was used by Belavin, Polyakov and Zamoldchicov (1984) to classify CFT's and find explicit expressions for the correlation functions in several cases

In more than 2 dimensions bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model) by Rychkov and others.

Conformal invariance

Scaling fields $V_\Delta(x)$, $x \in \mathbb{R}^d$, $\Delta \in \mathbb{R}$

Correlation functions invariant under rotations and translations of $\mathbb{R}^d$ and under scaling

$$\langle \prod_i V_{\Delta_i}(\lambda x_i) \rangle = \prod_i \lambda^{-2\Delta_i} \langle \prod_i V_{\Delta_i}(x_i) \rangle \quad (*)$$

Δ_i scaling dimension or **conformal weight**.

Conformal invariance: $(*)$ extends to conformal maps $x \to \Lambda(x)$,

E.g. in $d = 2$: $\mathbb{R}^2 \simeq \mathbb{C}$

$$\Lambda(z) = \frac{az+b}{cz+c} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

and $\lambda^{-2\Delta_i} \to |\Lambda'(z)|^{-2\Delta_i}$.

Natural setup is the **Riemann sphere**: $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Structure Constants

Use conformal map to fix three points to $\{0, 1, \infty\}$.

3-point functions are determined up to constants

$$\langle \prod_{k=1}^{3} V_{\Delta_k}(z_k) \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C(\Delta_1, \Delta_2, \Delta_3)$$

with $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc.

$$C(\Delta_1, \Delta_2, \Delta_3) = \langle V_{\Delta_1}(0) V_{\Delta_2}(1) V_{\Delta_3}(\infty) \rangle$$

are called the **structure constants** of the CFT.

Bootstrap hypothesis

Operator Product Expansion Axiom:

$$V_{\Delta_1}(x_1)V_{\Delta_2}(x_2) = \sum_{\Delta\in\mathcal{S}} C^{\Delta}_{\Delta_1\Delta_2}(x_1, x_2, \partial_{x_2})V_{\Delta}(x_2)$$

a **convergent** sum assumed to hold when inserted to expectation:

$$\langle V_{\Delta_1}(x_1)V_{\Delta_2}(x_2)V_{\Delta_3}(x_3)\dots\rangle = \sum_{\Delta\in\mathcal{S}} C^{\Delta}_{\Delta_1\Delta_2}(x_1, x_2, \partial_{x_2})\langle V_{\Delta}(x_2)V_{\Delta_3}(x_3)\dots\rangle$$

- $C^{\Delta}_{\Delta_1\Delta_2}$ are **determined** by and **linear** in the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- All correlations are determined by $C(\Delta_1, \Delta_2, \Delta_3)$

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

Examples in 2d

In **two dimensions** many explicit conjectures of spectra and structure constants exist:

- $\mathcal{S}$ is **finite**: minimal models (e.g. Ising model) (BPZ 1983)
- $\mathcal{S}$ is **countable**: Compact G WZW models, G/H coset theories
- $\mathcal{S}$ is **continuous**: **Liouville model**

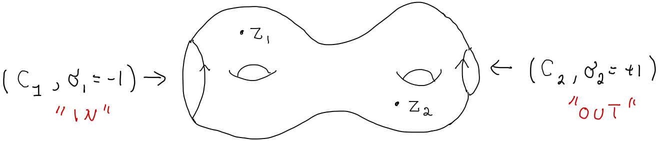
Segal's axioms

G. Segal (1987): 2d CFT is a **functor**: *Circle* $\to$ *Hilbert*

- Objects of *Circle* are $\sqcup_i C_i$, C_i is a (topological) circle
- Morphisms of *Circle* are closed oriented Riemann surfaces Σ with $n \geq 0$ marked points $z_1, \ldots, z_n$ and

  $$\partial\Sigma = \cup_i C_i$$

  together with analytic parametrisations $\zeta_i : \mathbb{T} \to C_i$.
- Set $\sigma_i = \pm 1$ depending on whether orientation of $\zeta_i(\mathbb{T})$ agrees with that of Σ or not. Call them "in" and "out" boundaries.



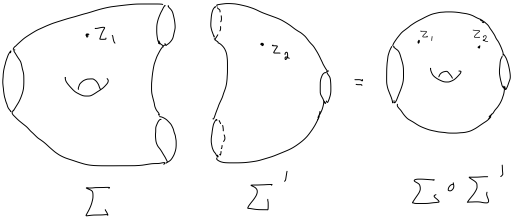
Segal's axioms

- Objects of *Hilbert* are $\mathcal{H}^{\otimes n}$, $\mathcal{H}$ Hilbert space
- Morphisms of *Hilbert* are Hilbert-Schmidt operators $\mathcal{A} : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$
- CFT functor maps $\sqcup_{i=1}^{n} C_i \to \mathcal{H}^{\otimes n}$ and $\Sigma \to \mathcal{A}_\Sigma$ with

$$\mathcal{A}_\Sigma : \otimes_{i \in I_{in}} H \to \otimes_{i \in I_{out}}$$

  such that

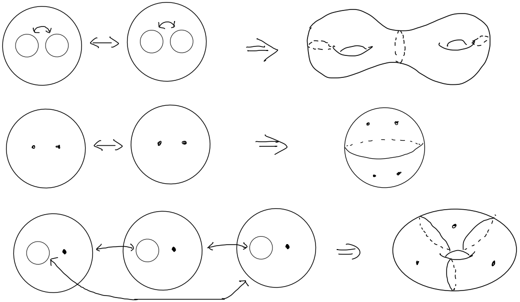
$$\mathcal{A}_{\Sigma \circ \Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$



Bootstrap

Build Σ by gluing simple building blocks $\mathcal{B}$:

- Pairs of pants $\mathcal{P} \sim \hat{\mathbb{C}} \setminus 3 \text{ disks}$
- Cylinders with one marked point $\hat{\mathbb{C}} \setminus \{2 \text{ disks}, \ 1 \text{ point}\}$
- Disks with two marked points $\hat{\mathbb{C}} \setminus \{1 \text{ disk}, \ 2 \text{ points}\}$



Correlation function on Σ is then given by compositions of $\mathcal{A}_{\mathcal{B}}$

Path integrals

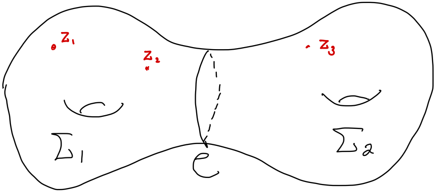
Motivation for axioms: let the QFT be given formally as a path integral, e.g. for a scalar field $\phi : \Sigma \to \mathbb{R}$, $\partial\Sigma = \emptyset$

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(z_i) \rangle = \int \prod_{i=1}^{n} W_i(\phi(z_i)) e^{-S_\Sigma(\phi)} D\phi_\Sigma$$

with

- Local action functional $S_\Sigma(\phi) = \int_\Sigma L(\phi(x), d\phi(x))$
- Formal Lebesgue measure $D\phi_\Sigma = \prod_{x\in\Sigma} d\phi(x)$

Let $\Sigma = \Sigma_1 \circ \Sigma_2$, $\partial\Sigma_i = \mathcal{C}$ so that $S_\Sigma = S_{\Sigma_1} + S_{\Sigma_2}$.



Path integrals

Define formally for $\varphi : \mathcal{C} \to \mathbb{R}$

$$\mathcal{A}_{\Sigma_j}(\varphi) = \int_{\phi|_{\Sigma_j}=\varphi} \prod_{i:z_i\in\Sigma_j} W_i(\phi(z_i))e^{-S_{\Sigma_j}(\phi)} D\phi_{\Sigma_j} \quad j = 1, 2$$

Then formally get

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(z_i)\rangle = \int_{\varphi:\mathcal{C}\to\mathbb{R}} \mathcal{A}_{\Sigma_1}(\varphi)\mathcal{A}_{\Sigma_2}(\varphi) D\varphi$$

Plan:

- Probabilistic construction of $\mathcal{A}_\Sigma$ for **Liouville CFT**
- Prove gluing $\mathcal{A}_{\Sigma\circ\Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$
- Use this to prove bootstrap and compute correlations.

Axioms for Weyl and Diff(Σ)

Two views of Riemann surfaces Σ

- Surface Σ with **complex structure**
- Surface Σ with **Riemannian metric** g

Moduli space of Riemann surfaces = **conformal classes** of metrics

$$g \sim e^{\sigma}\psi^{*}g \quad \psi \in Diff(\Sigma), \;\; \sigma \in C^{\infty}(\Sigma)$$

Diffeomorphism covariance axiom

$$\langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g} = \langle \prod_i V_{\Delta_i}(\psi(x_i)) \rangle_{\Sigma,\psi^* g}$$

Weyl covariance axiom:

$$\langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma,e^{\sigma}g} = e^{\frac{c}{96\pi}\int_{\Sigma}(\|d_g\varphi\|^2 + 2R_g\varphi)dv_g} \prod_i e^{-\Delta_i\sigma(x_i)} \langle \prod_{\alpha_i} V_i(x_i) \rangle_{\Sigma,g}$$

c **central charge** of the CFT.

$\implies g \to \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g}$ defined on **moduli space** of surfaces.

Classical Liouville Theory

Action functional

$$S(\phi, g) = \int_\Sigma (\|d\phi\|_g^2 + QR_g\phi + \mu e^{\gamma\phi}) dv_g$$

R_g scalar curvature, v_g volume measure.

- Uniformisation of Riemann surfaces (Picard, Poincaré1890's)
- $Q = \frac{2}{\gamma}$
- Minimizer ϕ_0 gives metric

  $$g_0 = e^{\gamma\phi_0} g$$

  with **constant (negative) curvature** R_{g_0}.
- μ, γ can be scaled to 1.

Quantum Liouville Theory

Path integral

$$\langle F \rangle = \int F(\phi) e^{-\int_\Sigma (\|d\phi\|_g^2 + QR_g\phi + \mu e^{\gamma\phi}) dv_g} D\phi$$

- Noncritical string theory (Polyakov 1981)
- 2d gravity Knizhnik, Polyakov, Zamolodchikov (1988)
- 4d SuSy Yang-Mills (Alday, Gaiotto, Tachikawa 2010)
- $e^{\gamma\phi}$ requires renormalisation and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.
- μ can be scaled to 1 by $\phi \to \phi + const$, γ only parameter.
- Conformal Field Theory with $c = 1 + 6Q^2$.

Probablistic Liouville Theory

We define

$$\langle F\rangle_{\Sigma,g} = Z_g \int_{\mathbb{R}} \mathbb{E}\big(F(\phi_g)e^{-\int_\Sigma (QR_g\phi_g+\mu:e^{\gamma\phi_g}:)dv_g}\big)dc$$

- $\phi_g = c + X_g$, X_g is Gaussian free field on (Σ, g)
- $: e^{\gamma\phi_g(z)} := e^{\gamma c}e^{\alpha X_g(z)-\frac{\gamma^2}{2}\mathbb{E}X_g(z)^2}$
- $: e^{\gamma\phi} : dv_g$ is the **Gaussian multiplicative chaos** measure
- $Z_g = (\det'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2}$ "partition function of GFF" defined by a zeta function.

Primary fields are **vertex operators**

$$V_\alpha(z) =: e^{\alpha\phi_g(z)} :$$

$: e^{\gamma\phi_g(z)} :$ and $: e^{\alpha\phi_g(z)} :$ are defined as limits of regularised objects.

Existence

Theorem (David, K, Rhodes, Vargas, 2015) *The Liouville correlation functions*

$$\langle \prod_i V_\alpha(z_i) \rangle_{\Sigma,g}$$

exist and are nontrivial if and only if the ***Seiberg bounds*** *hold:*

$$(1) \quad \alpha_i < Q \;\; \forall i, \quad \text{and} \quad (2) \;\; \sum_{i=1}^{n} \alpha_i + \chi(\Sigma) Q > 0$$

V_α are primary fields with scaling dimension $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ and the Weyl and Diffeo axioms hold.

Proof (2): convergence of c-integral

(1): regularity of GMC

Structure constants

For the structure constants we take $\Sigma = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81) Let α_i satisfy the Seiberg bounds. Then

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}} = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

where $C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ is an explicit formula conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov im 1995.

Proof combines **probabilistic** analysis of GMC to derive **algebraic** identities for the structure constants that determine them uniquely (Teschner).

Amplitudes

How to make sense of

$$\mathcal{A}_\Sigma(\varphi) = \int_{\phi|_\Sigma=\varphi} \prod_i V_{\alpha_i}(z_i) e^{-\int_\Sigma (\|d\phi\|_g^2 + QR_g\phi + \mu :e^{\gamma\phi}:) dv_g} D\phi_\Sigma ?$$

Consider the quadratic part

$$S^0(\phi) := \int_\Sigma \|d\phi\|_g^2 dv_g, \quad \phi|_{\partial\Sigma} = \varphi$$

Let $P\varphi$ be harmonic extension of φ:

$$\Delta_g \varphi = 0. \quad P\varphi|_{\partial\Sigma} = \varphi$$

By Green formula

$$S^0(\phi) = S^0(P\varphi) + S^0(Z), \quad Z = \phi - P\varphi$$

So since $Z|_{\partial\Sigma} = 0$ we should have

$$\mathcal{A}^0_\Sigma(\varphi) = e^{-S^0(P\varphi)} \mathcal{A}^0_\Sigma(0)$$

Free field amplitudes

We have again by a Green formula

$$S^0(P\varphi) = (\varphi, D_\Sigma \varphi)$$

where D_Σ is the **Dirichlet-Neumann** operator acting in $L^2(\mathbb{T}^n)$ for n boundary circles. We have

$$S^0(P\varphi) = \frac{1}{4}\sum_{n\in\mathbb{Z}} |n||\hat{\varphi}_n|^2 + (\varphi, \tilde{D}_\Sigma \varphi)$$

where $\tilde{D}_\Sigma$ is a smoothing operator defined on $H^{-s}(\mathbb{T}^n)$ $\forall s$.

Z is the Dirichlet GFF on Σ so we formally we have

$$\mathcal{A}^0_\Sigma(0) = \det(-\Delta_g^{dir})^{-\frac{1}{2}}.$$

Definition. The free field amplitude is defined by

$$\mathcal{A}^0_\Sigma(\varphi) = \det(-\Delta_g^{dir})^{-\frac{1}{2}} e^{(\varphi, \tilde{D}_\Sigma \varphi)}$$

where the determinant is zeta function regularised.

Liouville Amplitudes

Definition. The Liouville field amplitude with vertex operators at z_i

$$\mathcal{A}_\Sigma(\varphi) = \mathcal{A}^0_\Sigma(\varphi)\mathbb{E}\big(\prod V_{\alpha_i}(z_i)e^{-\int_\Sigma (QR_g\phi + \mu :e^{\gamma\phi}:)dv_g}\big)$$

where $\phi = P\varphi + Z$, and $\mathbb{E}$ is over the Dirichlet GFF Z.

Let μ be the measure on $\varphi = \sum_{n\in\mathbb{Z}} \varphi_n e^{in\theta} \in H^s(\mathbb{T})$, $s < 0$

$$d\mu(\varphi) = d\varphi_0 \prod_{n>0} e^{-|n|\,|\hat{\varphi}_n|^2} d\varphi_n d\varphi_{-n}$$

Take as the Segal Hilbert space for Liouville theory

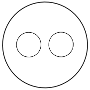
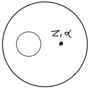
$$\mathcal{H} = L^2(H^s(\mathbb{T}), d\mu).$$

Then

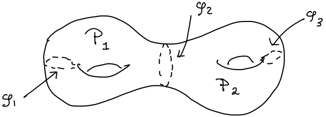
Proposition (GKRV'21). $\mathcal{A}_\Sigma$ are Hilbert-Schmidt opeartors and

$$\mathcal{A}_{\Sigma\circ\Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$

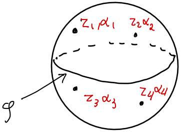
Examples

Building blocks: $\mathcal{P}$  $(\mathcal{C}, z, \alpha)$ 

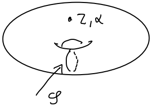
$(\mathbb{D}, \underset{\sim}{z}, \underset{\sim}{\alpha})$ 



$$\langle 1 \rangle_{\Sigma} = \int A_{\mathcal{P}_1}(\varphi_1, \varphi_1, \varphi_2) A_{\mathcal{P}_2}(\varphi_2, \varphi_3, \varphi_3) \prod_{i=1}^{3} d\mu(\varphi_i)$$



$$\left\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle_{\hat{\mathbb{C}}} = \int A_{\mathbb{D}, z_1 z_2 \alpha_1 \alpha_2}(\varphi) A_{\mathbb{D}, z_3 z_4 \alpha_3 \alpha_4}(\varphi)\, d\mu(\varphi)$$



$$\langle V_\alpha(z) \rangle_{\mathbb{T}^2} = \int A_{\mathcal{C}, z, \alpha}(\varphi, \varphi)\, d\mu(\varphi)$$

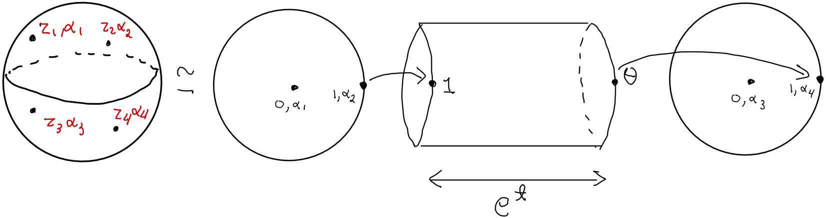
Plancharel

Let Ψ_i, $i \in \mathbb{N}$ be an orthonormal basis in $\mathcal{H}$. Then

$$\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \rangle_{\hat{\mathbb{C}}} = \sum_i (\mathcal{A}_{\mathbb{D}, z_1, z_2, \alpha_1, \alpha_2}, \Psi_i)(\Psi_i, \mathcal{A}_{\mathbb{D}, z_3, z_4, \alpha_3, \alpha_4})$$

What basis to use?

By Möbius $(\hat{\mathbb{C}}, z_1, z_2, z_3, z_4)$ has one modulus: encode it to a cylinder with a twist



Semigroup of cylinders

Let $q = e^{-t+i\theta}$ and $\mathcal{C}_q$ be a cylinder of length e^t, twist $e^{i\theta}$ in boundary parametrisations and $\mathcal{A}_q$ be its amplitude. Then

$$\mathcal{C}_q \circ \mathcal{C}_{q'} = \mathcal{C}_{qq'} \implies \mathcal{A}_q \mathcal{A}_{q'} = \mathcal{A}_{qq'}$$

Hence $q \to \mathcal{A}_q$ is a **semigroup** of operators in $\mathcal{H}$. It has two commuting self adjoint generators

$$\mathcal{A}_q = e^{-tH} e^{i\theta\Pi}$$

- $H \geq 0$ is the **Liouville Hamiltonian**
- $e^{i\theta\Pi}$ implements rotations of $\mathbb{T}$ in $\mathcal{H}$

Spectrum of Liouville theory

Theorem (GKRV 2020) $\mathcal{A}_q$ has a continuous spectrum and a complete set of generalised eigenfunctions $\Psi_{P,\nu,\tilde{\nu}}$:

$$\mathcal{A}_q \Psi_{P,\nu,\tilde{\nu}} = q^{\Delta_{Q+iP}+|\nu|} \bar{q}^{\Delta_{Q+iP}+|\tilde{\nu}|} \Psi_{P,\nu,\tilde{\nu}}$$

where $P \in \mathbb{R}$ and ν, $\tilde{\nu}$ are Young diagrams, $|\nu| = \sum \nu_i$.

Remarks.

1. $\Psi_{P,\nu,\tilde{\nu}}$ are related to representation theory: LCFT carries a representation of two commuting **Virasoro algebras**

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

with generators L_n, $\tilde{L}_n$, $n \in \mathbb{Z}$ and central charge $c = 1 + 6Q^2$. For $\nu = (\nu_1, \ldots, \nu_k)$, $\nu_i \geq \nu_{i+1}$

$$\Psi_{P,\nu,\tilde{\nu}} = L_{\nu_1} \ldots L_{\nu_k} \tilde{L}_{\tilde{\nu}_1} \ldots \tilde{L}_{\tilde{\nu}_k} \Psi_{P,0,0}$$

2. Think of $\Psi_{P,0,0}$ as the state in $\mathcal{H}$ produced by $V_{Q+iP}(0)$.

Holomorphic factorisation

Dependence on $\nu, \tilde{\nu}$ factorises also for the building blocks:

$$(\mathcal{A}_{\mathbb{D},0,1,\alpha_1,\alpha_2}, \Psi_{P,\nu,\tilde{\nu}}) = D(\alpha_1, \alpha_2, \nu) D(\alpha_1, \alpha_2, \tilde{\nu}) (\mathcal{A}_{\mathbb{D},0,1,\alpha_1,\alpha_2}, \Psi_{P,0,0})$$

Furthermore $(\mathcal{A}_{\mathbb{D},0,1,\alpha_1,\alpha_2}, \Psi_{P,0,0})$ is **3-point function on $\hat{\mathbb{C}}$**:

$$(\mathcal{A}_{\mathbb{D},0,1,\alpha_1,\alpha_2}, \Psi_{P,0,0}) = C_{DOZZ}(\alpha_1, \alpha_2, Q + iP)$$

Theorem. (GKRV 2020)

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(q) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}}} = \int_{\mathbb{R}} |q|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}(\boldsymbol{\alpha}, P, q)|^2$$
$$\times C_{DOZZ}(\alpha_1, \alpha_2, Q + iP) C_{DOZZ}(\alpha_3, \alpha_4, Q - iP) dP$$

$\mathcal{F}$ are **holomorphic conformal blocks**

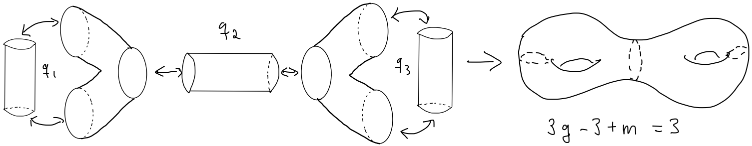
This calculation extends to all surfaces and correlation functions.

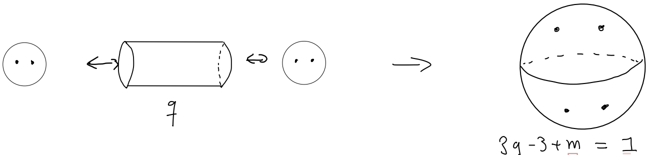
Plumbing

The moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g and n marked points is a complex orbifold of dimension $3g-3+n$.

$\mathcal{M}_{g,n}$ can be parametrised by (Hinich-Vaintrob 2011)

- Finite set of building blocks $\mathcal{B}_i$, $i = 1, \ldots N(g,n)$ where each $\mathcal{B}_i$ is a sphere with k punctures and $3-k$ boundary circles, $k = 0, 1, 2$.
- **Plumbing parameters $\mathbf{q} \in \mathbb{D}^{3g-3+n}$.**





General bootstrap

GKRV (2021). Let Σ have genus g. Then

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_\Sigma = \int_{\mathbb{R}_+^{3g+n-3}} |\mathcal{F}(\mathbf{q}, \mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

where

- $\mathbf{q}$ are plumbing parameters
- Conformal block $\mathcal{F}(\mathbf{q}, \mathbf{P})$ is holomorphic in $\mathbf{q}$
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha, \alpha', \alpha'')$ with $\alpha, \alpha', \alpha'' \in \{\alpha_i, Q \pm iP_j\}$

Open questions

Suppose Σ is parametrised by $(\{\mathcal{B}_i\}, \mathbf{q})$ and $(\{\mathcal{B}'_i\}, \mathbf{q}')$. Are the blocks linearly related? True for $(g, n) = (0, 4)$ and $(g, n) = (1, 1)$.

Connection to quantisation of Teichmuller space?

Thank you!